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CONTENTS

Parameters Identification of Thermoelastic Problems for Compound Hollow Sphere under Nonstationary Temperature Field

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ABSTRACT

We constructed explicit expressions of gradients of functional-discrepancies for identification by gradient methods of different parameters of problems of thermoelastic deformation of compound hollow sphere under nonstationary field of temperature. Gradients were constructed on the basis of theory of optimal control of states of multicomponent distributed systems.

Key words: explicit expressions, gradients of functional-discrepancies, identification, gradient methods, parameters of problems of thermoelastic deformation, compound hollow sphere, nonstationary field of temperature, theory of optimal control of states.

In publications [1–3] on the basis of results of theory of optimal control [4–6] we obtained explicit expressions of gradients of functional-discrepancies for identification by the O.M. Alifanov gradient methods [7] of different parameters of problems of elastic deformation of compound hollow sphere, compound cylinder and thermoelastic deformation of compound spatial body correspondingly.

In the present article similar questions are considered for identification of different parameters of problems of thermoelatic deformation of compound hollow sphere under nonstationary field of temperature.

1. Identification of thermoelastic state by surface displacements

Let us consider isotropic hollow sphere. Taking into account symmetry, following [8, 9] its thermostressed state under assumption about smallness of inertia terms $\rho \ddot{y}$ (y is radial displacement) is described by the equation

$$\frac{\partial \sigma_r(y)}{\partial r} + \frac{2\sigma_r(y) - \sigma_\theta(y) - \sigma_\varphi(y)}{r} = 0, \ (r, t) \in \Omega_T, \tag{1}$$

where $\Omega_T = \Omega \times (0, \overline{T})$, $\Omega = (r_1, r_2)$, $0 < r_1 < r_2 < \infty$, $(0, \overline{T})$ is time interval,

$$\sigma_r = (\lambda + 2\mu)\frac{\partial y}{\partial r} + 2\lambda \frac{y}{r} - (3\lambda + 2\mu)\alpha T,$$

$$\sigma_\varphi = \sigma_\theta = \lambda \frac{\partial y}{\partial r} + \frac{2(\lambda + \mu)}{r} y - (3\lambda + 2\mu)\alpha T. \tag{1$'$}$$

Here λ, μ are the Lame elastic constants, α is the linear expansion coefficient, T is variation of temperature $\overline{T}_1$ from its initial state $\overline{T}_{10}$.

Taking into account (1$'$) the equality (1) is easily transformed to the form

$$-\left\{(\lambda + 2\mu)\left(\frac{\partial}{\partial r}\left(r^2 \frac{\partial y}{\partial r}\right) - 2y\right) - (3\lambda + 2\mu)\alpha r^2 \frac{\partial T}{\partial r}\right\} = 0, \ (r, t) \in \Omega_T. \tag{2}$$

Variation of temperature T holds the equation

$$c\frac{\partial T}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 k \frac{\partial T}{\partial r}\right) + \bar{f}, \ (r, t) \in \Omega_T, \tag{3}$$

where c is volumetric heat capacity, k is heat conductivity factor, $\bar{f}$ is power of sources of heat.

On internal and external surfaces of sphere the stresses are given

$$\sigma_r(y)\big|_{r=r_i} = -p_i, \ i = 1, 2, \ t \in (0, \overline{T}). \tag{4}$$

Density of heat flow on internal surface is

$$-k\frac{\partial T}{\partial r} = u, \ r = r_1, \ t \in (0, \overline{T}), \tag{5}$$

which is supposed to be unknown, and on external surface boundary condition of the third kind is set

$$k\frac{\partial T}{\partial r} = -\overline{\alpha} T + \beta, \ r = r_2, \ t \in (0, \overline{T}). \tag{6}$$

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At initial time instant distribution of variation of temperature field is known

$$T\big|_{t=0} = \overline{T}_0, \ r \in \overline{\Omega}. \tag{7}$$

We assume that on external surface of sphere its displacement is known

$$y(r_2, t) = f_0(t), \ t \in (0, \overline{T}). \tag{8}$$

In this way we obtained the problem (2)–(8), which consists in determination of an element $u \in \mathcal{U} = C([0, \overline{T}])$, for which the first component of y of the classical solution $Y = (y, T)$ of the initial boundary value problem (2)–(7) holds the equality (8).

Let us introduce into consideration the functional-discrepancy

$$J(u) = \frac{1}{2} \int\limits_0^{\overline{T}} \| Au - f_0 \|^2 dt, \tag{9}$$

where $Au = y(u; r_2, t)$, $\| Au - f_0 \| = |Au - f_0|$.

For every fixed $u \in \mathcal{U}$ instead of classical solution $Y = (y, T)$ of the initial boundary value problem (2)–(7) we shall use its generalized solution.

Definition 1. For every fixed $u \in \mathcal{U}$ we call as the generalized solution of the initial boundary value problem (2)–(7) the vector-function $Y = Y(u) = (y(u), T(u)) \in V$, which $\forall z = (z_1, z_2) \in V_0$ holds the system of relations

$$a(y, z_1) = l(T; z_1), \ t \in (0, \overline{T}), \tag{10}$$

$$\left(r^2 c \frac{\partial T}{\partial t}, z_2 \right) + a_1(T, z_2) = l_1(u; z_2), \ t \in (0, \overline{T}), \tag{11}$$

$$(r^2 c T, z_2)(0) = (r^2 c \overline{T}_0, z_2), \tag{12}$$

where

$$V = \left\{ v = (v_1(r, t), v_2(r, t)) : v_i \in W_2^1(\Omega), \frac{\partial v_2}{\partial t} \in L^2(0, \overline{T}; L_2(\Omega)), i = 1, 2; \right.$$

$$\left. \forall t \in [0, \overline{T}], \int\limits_0^{\overline{T}} \sum_{i=1}^{2} \| v_i \|_{W_2^1(\Omega)}^2 dt < \infty \right\},$$

$$V_0 = \{ v(r) = (v_1(r), v_2(r)) : v_i \in W_2^1(\Omega), i = 1, 2 \},$$

$$a(y, z_1) = \int\limits_{r_1}^{r_2} r^2 \left\{ (\lambda + 2\mu) \left(\frac{\partial y}{\partial r} \frac{\partial z_1}{\partial r} + 2 \frac{y}{r} \frac{z_1}{r} \right) + 2\lambda \left(\frac{\partial y}{\partial r} \frac{z_1}{r} + \frac{y}{r} \frac{\partial z_1}{\partial r} + \frac{y}{r} \frac{z_1}{r} \right) \right\} dr,$$

$$a_1(T, z_2) = \int\limits_{r_1}^{r_2} r^2 k \frac{\partial T}{\partial r} \frac{\partial z_2}{\partial r} dr + r_2^2 \overline{\alpha} T(r_2, t) z_2(r_2),$$

$$l(T; z_1) = \int\limits_{r_1}^{r_2} r^2 (3\lambda + 2\mu) \alpha T \left(\frac{\partial z_1}{\partial r} + \frac{2z_1}{r} \right) dr + r_1^2 p_1 z_1(r_1) - r_2^2 p_2 z_1(r_2),$$

$$l_1(u; z_2) = \int\limits_{r_1}^{r_2} r^2 \overline{f} z_2 dr + r_1^2 u z_2(r_1) + r_2^2 \beta z_2(r_2).$$

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We shall solve the problem (10)–(12), (9), which consists in determination of an element u minimizing on $\mathcal{U}$ the functional (9) under the constraints (10)–(12), approximately by means of gradient methods [7]. Iteration sequence for determination of the $(n+1)$-th approximation u_{n+1} of solution $u \in \mathcal{U}$ of the problem (9)–(12) has the form

$$u_{n+1} = u_n - \beta_n p_n. \tag{13}$$

It starts from certain initial approximation $u_0 \in \mathcal{U}$, where direction of descent p_n and the coefficient β_n are determined by the expressions [7]

— for the method of minimal errors

$$p_n = J'_{u_n}, \quad \beta_n = \frac{\|e_n\|^2}{\|J'_{u_n}\|^2}; \tag{14}$$

— for the method of steepest descent

$$p_n = J'_{u_n}, \quad \beta_n = \frac{\|J'_{u_n}\|^2}{\|AJ'_{u_n}\|^2}; \tag{15}$$

— for the method of conjugate gradients

$$p_n = J'_{u_n} + \gamma_n p_{n-1}, \quad \gamma_0 = 0, \quad \gamma_n = \frac{\|J'_{u_n}\|^2}{\|J'_{u_{n-1}}\|^2}, \quad \beta_n = \frac{(J'_{u_n}, p_n)}{\|Ap_n\|^2}, \tag{16}$$

where J'_u is gradient of the functional (9) at the point $u = u_n$, $e_n = Au_n - f_0$, $Au_n = y(u_n; r_2, t)$.

Let us introduce into consideration denotations

$$\pi(u, v) = (\bar{y}(u) - \bar{y}(0), \ \bar{y}(v) - \bar{y}(0))_{L_2},$$
$$L(v) = (f_0 - \bar{y}(0), \ \bar{y}(v) - \bar{y}(0))_{L_2}, \tag{17}$$

where $\forall v \in \mathcal{U}$ $\bar{y}(v) = y(v; r_2, t)$, $y(v; r_2, t)$ is the first component of the solution $Y = Y(v) = (y(v), T(v))$ of the problem (10)–(12) for $u = v$, $(\bar{\varphi}, \bar{\psi})_{L_2} = \int\limits_0^{\bar{T}} \bar{\varphi}\bar{\psi}\, dt$.

The following equality takes place:

$$2J(v) = \pi(v, v) - 2L(v) + \|f_0 - \bar{y}(0)\|^2_{L_2}. \tag{18}$$

Let $u, v \in \mathcal{U}$. For $\lambda \in (0, 1)$ $z = \lambda v + (1 - \lambda)u = u + \lambda(v - u) \in \mathcal{U}$. Taking into account (17), (18) we obtain

$$\lim_{\lambda \to 0} \frac{J(u + \lambda(v - u)) - J(u)}{\lambda} = \pi(u, v - u) - L(v - u) =$$
$$= (\bar{y}(u) - f_0, \ \bar{y}(v) - \bar{y}(u))_{L_2} = \langle J'_u, v - u \rangle. \tag{19}$$

For every approximation u_n of solution $u \in \mathcal{U}$ of the problem (9)–(12) following [1–3, 10] we introduce into consideration the following conjugate problem:

$$-(\lambda + 2\mu)\left(\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) - 2\psi\right) = 0, \quad (r, t) \in \Omega_T,$$

$$\sigma_r(\psi)\big|_{r=r_1} = 0, \ \sigma_r(\psi)\big|_{r=r_2} = \frac{1}{r_2^2}(y(u_n; r_2, t) - f_0), \ t \in (0, \overline{T}),$$

$$-r^2 c \frac{\partial p}{\partial t} - \frac{\partial}{\partial r}\left(r^2 k \frac{\partial p}{\partial r}\right) - r^2 (3\lambda + 2\mu)\alpha(\varepsilon_r(\psi) + \varepsilon_\varphi(\psi) + \varepsilon_\theta(\psi)) = 0, \ (r, t) \in \Omega_T, \tag{20}$$

$$-k\frac{\partial p}{\partial r}\bigg|_{r=r_1} = 0, \ k\frac{\partial p}{\partial r}\bigg|_{r=r_2} = -\overline{\alpha}\, p(r_2, t), \ t \in (0, \overline{T}),$$

$$p\big|_{t=\overline{T}} = 0, \ r \in \Omega,$$

where $\Omega_T = \Omega \times (0, \overline{T})$, $\sigma_r(\psi) = (\lambda + 2\mu)\dfrac{\partial\psi}{\partial r} + 2\lambda\dfrac{\psi}{r}$, $\varepsilon_r(\psi) = \dfrac{\partial\psi}{\partial r}$, $\varepsilon_\varphi(\psi) = \varepsilon_\theta(\psi) = \dfrac{\psi}{r}$.

Definition 2. We call as the generalized solution of the initial boundary value problem (20) the vector-function $Y^* = (\psi, p) \in V$, which $\forall z = (z_1, z_2) \in V_0$ holds the system of relations:

$$a(\psi, z_1) = (y(u_n; r_2, t) - f_0)z_1(r_2), \ t \in (0, \overline{T}), \tag{21}$$

$$-\left(r^2 c \frac{\partial p}{\partial t}, z_2\right) + a_1(p, z_2) - \int_\Omega r^2 (3\lambda + 2\mu)\alpha z_2\left(\frac{\partial\psi}{\partial r} + \frac{2\psi}{r}\right) dr = 0, \ t \in (0, \overline{T}), \tag{22}$$

$$(r^2 cp, z_2)(\overline{T}) = 0. \tag{23}$$

If we substitute in (21) z_1 by the difference $y(u_{n+1}) - y(u_n)$ and z_2 in (22), (23) by the difference $T(u_{n+1}) - T(u_n)$, taking into account (10)–(12) we obtain

$$\int_0^{\overline{T}} (y(u_n; r_2, t) - f_0)(y(u_{n+1}; r_2, t) - y(u_n; r_2, t))\, dt = \int_0^{\overline{T}} a(\psi, y(u_{n+1}) - y(u_n))\, dt +$$

$$+ \int_0^{\overline{T}} \left(r^2 c \frac{\partial(T(u_{n+1}) - T(u_n))}{\partial t}, p\right) dt + \int_0^{\overline{T}} a_1(p, T(u_{n+1}) - T(u_n))\, dt -$$

$$- \int_0^{\overline{T}}\int_\Omega r^2 (3\lambda + 2\mu)\alpha(T(u_{n+1}) - T(u_n))\left(\frac{\partial\psi}{\partial r} + \frac{2\psi}{r}\right) dr\, dt = \int_0^{\overline{T}} \Delta u_n r_1^2 p(r_1, t)\, dt,$$

i.e.,

$$\left\langle J'_{u_n}, \Delta u_n \right\rangle = (\Delta u_n r_1^2, p\big|_{r=r_1})_{L_2}. \tag{24}$$

Hence,

$$J'_{u_n} = \widetilde{\psi}_n, \tag{25}$$

where $\widetilde{\psi}_n = r_1^2 p(r_1, t)$, $\left\|\widetilde{\psi}_n\right\|^2 = \int_0^{\overline{T}} (\widetilde{\psi}_n)^2\, dt$.

The presence of gradient J'_{u_n} makes it possible to use gradient methods (13) for determination of the $(n+1)$-th approximation u_{n+1} of the desired solution $u \in \mathcal{U}$ of the problem (9)–(12).

Remark 1. If the recoverable heat flow u is assumed to be constant, i.e., if $\mathcal{U} = (-\infty, +\infty)$, then on the basis of (24) we obtain $J'_{u_n} = \widetilde{\psi}_n$, where $\widetilde{\psi}_n = \int_0^{\overline{T}} r_1^2 p(r_1, t)\, dt$, $\left\|J'_{u_n}\right\| = \left|\widetilde{\psi}_n\right|$.

Remark 2. If the recoverable heat flow $u = u(t)$ is assumed to be representable as

$$u = u_m(t) = \sum_{i=1}^{m} \alpha_i \varphi_i(t), \tag{26}$$

where $\{\varphi_i(t)\}_{i=1}^{m}$ is a system of linearly independent functions, then we obtain parametric technique for recovering the flow. On the basis of (24) we have

$$J'_{u_n} = \widetilde{\psi}_n, \quad \widetilde{\psi}_n = \{\widetilde{\psi}_n^i\}_{i=1}^{m}, \quad \widetilde{\psi}_n^i = \int_0^{\overline{T}} r_1^2 \varphi_i \, p(r_1, t) \, dt, \quad \left\| J'_{u_n} \right\|^2 = \sum_{i=1}^{m} (\widetilde{\psi}_n^i)^2. \tag{26$'$}$$

The presence of gradient J'_{u_n} makes it possible to use the method of minimal errors (13), (14) for determination of the $(n+1)$-th approximation $u_{n+1} = \{\alpha_i^{n+1}\}_{i=1}^{m}$ of the solution $u = \{\alpha_i\}_{i=1}^{m} \in \mathcal{U} = R^m$ of the problem (9)–(12), where recoverable flow u of the boundary condition (5) is searched in the form (26). In this case on determination of solution $Y(u_{n+1})$ of the problem (9)–(12)

$$l_1(u_{n+1}; z_2) = \int_{r_1}^{r_2} r^2 \overline{f} \, z_2 \, dr + r_1^2 \sum_{i=1}^{m} \alpha_i^{n+1} \varphi_i(t) \, z_2(r_1) + r_2^2 \beta z_2(r_2).$$

If we solve the problem of determination of the vector-function $Y = (y(J'_{u_n}), T(J'_{u_n}))$, which $\forall z = (z_1, z_2) \in V_0$ holds the system of relations

$$a(y, z_1) = l(T(J'_{u_n}); z_2), \quad t \in (0, \overline{T}),$$

$$\left(r^2 c \frac{\partial T}{\partial t}, z_2 \right) + a_1(T, z_2) = l_1(J'_{u_n}; z_2), \quad t \in (0, \overline{T}), \tag{27}$$

$$(r^2 c T, z_2)(0) = (r^2 c \overline{T}_0, z_2),$$

we obtain $A J'_{u_n} = y(J'_{u_n}; r_2, t)$, which makes it possible to use the method of steepest descent (13)–(15) for search of the $(n+1)$-th approximation u_{n+1} of solution of the problem (9)–(12), where

$$l_1(J'_{u_n}; z_2) = \int_{r_1}^{r_2} r^2 \overline{f} \, z_2 \, dr + r_1^2 J'_{u_n} z_2(r_1) + \beta r_2^2 z_2(r_2), \text{ if } u \in \mathcal{U} = C([0, \overline{T}]), \text{ and } l_1(J'_{u_n}; z_2) = \int_{r_1}^{r_2} r^2 \overline{f} \, z_2 \, dr +$$

$$+ r_1^2 \sum_{i=1}^{m} \widetilde{\psi}_n^i \varphi_i(t) z_2(r_1) + \beta r_2^2 z_2(r_2), \text{ if the recoverable parameter } u \text{ of the boundary condition (5) is searched in the form (26).}$$

If we determine the direction of descent p_n by means of the expressions (16), we can solve the problem of (27) type, where instead of J'_{u_n} we use p_n. This makes it possible to use the method of conjugate gradients (13), (16) for searching the $(n+1)$-th approximation u_{n+1} of solution $u \in \mathcal{U}$ of the problem (9)–(12).

2. Identification of thermostressed state of hollow sphere by displacements of its internal point

Let on every fixed $u \in \mathcal{U} = C([0, \overline{T}])$ thermostressed state of hollow sphere is described by the initial boundary value problem (2)–(7), i.e., the generalized problem (10)–(12). We assume that at the internal point $d_1 \in (r_1, r_2)$ displacement is known and given by the equality

$$y(d_1, t) = f_1(t), \quad t \in (0, \overline{T}). \tag{28}$$

In this case functional-discrepancy has the form

$$J(u) = \frac{1}{2} \int_0^{\overline{T}} (Au - f_1)^2 dt, \tag{29}$$

where $Au = y(u; d_1, t)$, $y(u; r, t)$ is the first component of solution $Y = Y(u) = (y(u), T(u))$ of the problem (10)–(12).

The expressions of (17)–(19) type take place, where $\overline{y}(v) = y(v; d_1, t)$, $y(v; r, t)$ the first component of solution $Y = (y, T)$ of the problem (10)–(12) for $u = v$.

For every approximation u_n of solution $u \in \mathscr{U}$ of the problem (10)–(12), (29) the conjugate problem has the form:

$$-(\lambda + 2\mu)\left(\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r} \right) - 2\psi \right) = 0, \ (r, t) \in \Omega_{dT},$$

$$\sigma_r(\psi)\big|_{r=r_i} = 0, \ i = 1, 2, \ t \in (0, \overline{T}),$$

$$[\psi]\big|_{d_1} = 0, \ [\sigma_r(\psi)]\big|_{d_1} = -\frac{1}{d_1^2}(y(u_n; d_1, t) - f_1(t)), \ t \in (0, \overline{T}),$$

$$-r^2 c \frac{\partial p}{\partial t} - \frac{\partial}{\partial r}\left(r^2 k \frac{\partial p}{\partial r} \right) - r^2 (3\lambda + 2\mu)\alpha\left(\frac{\partial \psi}{\partial r} + \frac{2\psi}{r} \right) = 0, \ (r, t) \in \Omega_{dT}, \tag{30}$$

$$[p]_{d_1} = 0, \ \left[k \frac{\partial p}{\partial r} \right]\bigg|_{d_1} = 0, \ t \in (0, \overline{T}),$$

$$-k \frac{\partial p}{\partial r}\bigg|_{r=r_1} = 0, \ k \frac{\partial p}{\partial r}\bigg|_{r=r_2} = -\overline{\alpha} p(r_2, t), \ t \in (0, \overline{T}),$$

$$p\big|_{\overline{T}} = 0, \ r \in \overline{\Omega},$$

where $\Omega_{dT} = (\Omega \setminus (r = d_1)) \times (0, \overline{T})$, the component $\sigma_r(\psi)$ is defined in section 1.

Definition 3. We call as the generalized solution of the initial boundary value problem (30) the vector-function $Y^* = (\psi, p) \in V_d$, which $\forall z = (z_1, z_2) \in V_d^0$ holds the system of relations

$$a(\psi, z_1) = (y(u_n; d_1, t) - f_1(t)) z_1(d_1), \ t \in (0, \overline{T}), \tag{31}$$

$$-\left(r^2 c \frac{\partial p}{\partial t}, z_2 \right) + \alpha_1(p, z_2) - \int_\Omega r^2 (3\lambda + 2\mu)\alpha z_2\left(\frac{\partial \psi}{\partial r} + \frac{2\psi}{r} \right) dr = 0, \ t \in (0, \overline{T}), \tag{32}$$

$$(r^2 cp, z_2)(\overline{T}) = 0, \tag{33}$$

where

$$V_d = \left\{ v = (v_1(r, t), v_2(r, t)) : v_i\big|_{\Omega_j} \in W_2^1(\Omega_j), [v_i]\big|_{r=d_1} = 0, i, j = 1, 2, \right.$$

$$\forall t \in [0, \overline{T}] \int_0^{\overline{T}} \sum_{j=1}^2 \|v_i\|_{W_2^1(\Omega_j)}^2 dt < \infty, \left. \frac{\partial v_2}{\partial t}\bigg|_{\Omega_i} \in L^2(0, \overline{T}; L_2(\Omega_i)), i = 1, 2 \right\},$$

$$V_d^0 = \{ v = (v_1(r), v_2(r)) : v_i\big|_{\Omega_j} \in W_2^1(\Omega_j), [v_i]\big|_{r=d_1} = 0, i, j = 1, 2 \}.$$

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If we select in the relation (31) instead of the function z_1 the difference $y(u_{n+1}) - y(u_n)$ and the difference $T(u_{n+1}) - T(u_n)$ instead of z_2 in (32), (33), taking into account (10)–(12) we obtain

$$\langle J'_{u_n}, \Delta u_n \rangle = \int\limits_0^{\overline{T}} (y(u_n; d_1, t) - f_1)(y(u_{n+1}; d_1, t) - y(u_n; d_1, t))\, dt =$$

$$= \int\limits_0^{\overline{T}} a(y(u_{n+1}) - y(u_n), \psi)\, dt - \int\limits_0^{\overline{T}}\int\limits_\Omega r^2 (3\lambda + 2\mu)\, \alpha\, (T(u_{n+1}) - T(u_n))\left(\frac{\partial \psi}{\partial r} + \frac{2\psi}{r}\right) dr\, dt +$$

$$+ \int\limits_0^{\overline{T}}\left(r^2 c\, \frac{\partial(T(u_{n+1}) - T(u_n))}{\partial t},\, p\right) dt + \int\limits_0^{\overline{T}} a_1(T(u_{n+1}) - T(u_n),\, p)\, dt =$$

$$= \int\limits_0^{\overline{T}} (l_1(u_{n+1}; p) - l_1(u_n; p))\, dt = \int\limits_0^{\overline{T}} \Delta u_n r_1^2 p(r_1, t)\, dt.$$

Hence, $J'_{u_n} = \widetilde{\psi}_n$, where $\widetilde{\psi}_n = r_1^2 p(r_1, t)$, $\left\| J'_{u_n} \right\|^2 = \int\limits_0^{\overline{T}} \widetilde{\psi}_n^2 dt$.

Remark 3. If instead of the condition (28) we have also the condition (8), then we write the functional-discrepancy as

$$J(u) = \frac{1}{2} \sum_{i=0}^1 \int\limits_0^{\overline{T}} (y(u; d_i, t) - f_i(t))^2\, dt,\ \ d_0 = r_2. \tag{34}$$

In this case we have the problem (10)–(12), (34).

Let us introduce denotations

$$\pi(u, v) = (\overline{y}(u) - \overline{y}(0),\, \overline{y}(v) - \overline{y}(0))_{L_2},$$

$$L(v) = (\overline{f} - \overline{y}(0),\, \overline{y}(v) - \overline{y}(0))_{L_2},$$

where $\overline{y}(v) = (y(v; d_0, t), y(v; d_1, t))$, $\overline{f} = (f_0, f_1)$, $\overline{\varphi} = (\varphi_1(t), \varphi_2(t))$, $\overline{\psi} = (\psi_1(t), \psi_2(t))$, $(\overline{\varphi}, \overline{\psi})_{L_2} =$

$$= \int\limits_0^{\overline{T}} \sum_{i=1}^2 \varphi_i(t)\psi_i(t)\, dt.$$

Since

$$2J(v) = \pi(v, v) - 2L(v) + \left\| \overline{f} - \overline{y}(0) \right\|_{L_2}^2,$$

then

$$\lim_{\lambda \to 0} \frac{J(u + \lambda(v - u)) - J(u)}{\lambda} = \pi(u, v - u) - L(v - u) =$$

$$= (\overline{y}(u) - \overline{f},\, \overline{y}(v) - \overline{y}(u))_{L_2} = \langle J'_u, v - u \rangle. \tag{35}$$

For every approximation u_n of solution $u \in \mathcal{U}$ of the problem (10)–(12), (34) the conjugate problem has the form (30), where instead of the second restriction, which reflects statement of boundary conditions, we accept

$$\sigma_r(\psi)\big|_{r=r_1} = 0,\ \ \sigma_r(\psi)\big|_{r=r_2} = \frac{1}{r_2^2}(y(u_n; r_2, t) - f_0(t)),\ \ t \in (0, \overline{T}).$$

For this initial boundary value problem the generalized problem consists in search of the vector-function $Y^* = (\psi, p) \in V_d$, which $\forall z = (z_1, z_2) \in V_d^0$ holds the relations

$$a(\psi, z_1) = \sum_{i=0}^{1} (y(u_n; d_i, t) - f_i(t)) z_1(d_i), \, t \in (0, \overline{T}),$$

$$-\left(r^2 c \frac{\partial p}{\partial t}, z_2\right) + a_1(p, z_2) - \int_\Omega r^2 (3\lambda + 2\mu) \alpha \left(\frac{\partial \psi}{\partial r} + \frac{2\psi}{r}\right) z_2 dr = 0, \ t \in (0, \overline{T}), \tag{36}$$

$$(r^2 cp, z_2)(\overline{T}) = 0.$$

Taking into account (35) on the basis of (36) we obtain $J'_{u_n} = \widetilde{\psi}_n = r^2 p(r_1, t)$.

3. Restoration of the linear expansion coefficient by surface displacements

For unknown linear expansion coefficient α the components $\sigma_r(y), \sigma_\varphi(y), \sigma_\theta(y)$ instead of $(1')$ take the form

$$\sigma_r(y) = (\lambda + 2\mu)\frac{\partial y}{\partial r} + 2\lambda \frac{y}{r} - (3\lambda + 2\mu) u T,$$

$$\sigma_\varphi(y) = \sigma_\theta(y) = \lambda \frac{\partial y}{\partial r} + \frac{2(\lambda + \mu)}{r} y - (3\lambda + 2\mu) u T, \tag{37}$$

where nonnegative real constant $u \in \mathcal{U} = [0, +\infty)$ is to be determined.

Taking into account (37) on the basis of (1) the equilibrium equation takes the form

$$-\left\{(\lambda + 2\mu)\frac{\partial}{\partial r}\left(r^2 \frac{\partial y}{\partial r}\right) - (3\lambda + 2\mu) u r^2 \frac{\partial T}{\partial r} - 2(\lambda + 2\mu) y\right\} = 0, \ (r, t) \in \Omega_T, \tag{38}$$

where $\Omega_T = \Omega \times (0, \overline{T})$, $\Omega = (r_1, r_2)$.

Variation of temperature T holds the equation (3). On internal and external surfaces of the hollow sphere the stresses are given (4), and variation of the temperature T holds the mixed boundary conditions

$$T(r_1, t) = \overline{u}_0(t), \ k\frac{\partial T}{\partial r}\bigg|_{r=r_2} = -\overline{\alpha} T\big|_{r=r_2} + \beta, \ t \in (0, \overline{T}). \tag{39}$$

For $t = 0$ the initial condition has the form

$$T(r, 0) = \overline{T}_0(r), \ r \in \Omega. \tag{40}$$

We assume that on external surface of the sphere the displacements, given by the equality (8), are known.

So, we obtained the problem (38)–(40), (3), (4), (8), which consists in determination of nonnegative number $u \in \mathcal{U}$, for which the first component y of the classical solution $Y = (y, T)$ of the initial boundary value problem (38)–(40), (3), (4) holds the equality (8).

For every fixed $u \in \mathcal{U}$ instead of the classical solution $Y = (y, T)$ of the initial boundary value problem (38)–(40), (3), (4) we shall use its generalized solution.

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Definition 4. For every fixed $u \in \mathcal{U}$ we call as the generalized solution of the initial boundary value problem (38)–(40), (3), (4) the vector-function $Y = (y, T) \in V$, which $\forall z = (z_1, z_2) \in V_0$ holds the system of relations:

$$a(y, z_1) = l(u, T; z_1), \ t \in (0, \overline{T}),$$

$$\left(r^2 c \frac{\partial T}{\partial t}, z_2 \right) + a_1(T, z_2) = l_1(z_2), \ t \in (0, \overline{T}), \tag{41}$$

$$(r^2 c T, z_2)(0) = (r^2 c \overline{T}_0, z_2),$$

where bilinear forms $a(\cdot, \cdot)$, $a_1(\cdot, \cdot)$ are defined in section 1,

$$l(u, T; z_1) = \int_{r_1}^{r_2} r^2 (3\lambda + 2\mu) u T \left(\frac{\partial z_1}{\partial r} + \frac{2 z_1}{r} \right) dr + r_1^2 p_1 z_1(r_1) - r_2^2 p_2 z_1(r_2),$$

$$l_1(z_2) = \int_{r_1}^{r_2} r^2 \overline{f} z_2 \, dr + \beta r_2^2 z(r_2),$$

$$V = \left\{ v = (v_1(r, t), v_2(r, t)) : v_i \big|_{\Omega} \in W_2^1(\Omega), \ v_2(r_1, t) = \overline{u}_0(t), \right.$$

$$\left. \forall t \in [0, \overline{T}], \int_0^{\overline{T}} \|v_i\|_{W_2^1(\Omega)}^2 \, dt < \infty, \ i = 1, 2, \ \frac{\partial v_2}{\partial t} \in L^2(0, \overline{T}; L_2(\Omega)) \right\},$$

$$V_0 = \{ v = (v_1(r), v_2(r)) : v_i \big|_{\Omega} \in W_2^1(\Omega), \ i = 1, 2, \ v_2(r_1) = 0 \}.$$

The functional-discrepancy has the form (9). We shall solve the problem (9), (41) by means of gradient methods (13).

For every approximation u_n of solution $u \in \mathcal{U}$ of the problem (9), (41) the conjugate problem has the form

$$-(\lambda + 2\mu) \left(\frac{\partial}{\partial r} \left(r^2 \frac{d\psi}{dr} \right) - 2\psi \right) = 0, \ (r, t) \in \Omega_T,$$

$$\sigma_r(\psi) \big|_{r=r_1} = 0, \ \sigma_r(\psi) \big|_{r=r_2} = \frac{1}{r_2^2} (y(u_n; r_2) - f_0), \ t \in (0, \overline{T}), \tag{42}$$

where the component $\sigma_r(\psi)$ is defined in section 1.

Definition 5. We call as the generalized solution of the boundary value problem (42) the function

$$\psi(r, t) \in V_1 = \left\{ v(r, t) : v \in W_2^1(\Omega) \ \forall t \in [0, \overline{T}], \int_0^{\overline{T}} \|v\|_{W_2^1(\Omega)}^2 \, dt < \infty \right\}, \text{ which } \forall z_1(r) \in V_1^0 = W_2^1(\Omega) \text{ holds}$$

the relation

$$a(\psi, z_1) = (y(u_n) - f_0) \big|_{r=r_2} z_1(r_2), \ t \in (0, \overline{T}). \tag{43}$$

If we substitute in (43) the function z_1 by the difference $y(u_{n+1}) - y(u_n)$, taking into account the first relation of the system (41) we obtain

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$$\int_0^{\overline{T}} (y(u_n) - f_0)(y(u_{n+1}) - y(u_n))\Big|_{r=r_2} dt = \int_0^{\overline{T}} a(\psi, y(u_{n+1}) - y(u_n)) dt =$$

$$= \int_0^{\overline{T}} (l(u_{n+1}, T; \psi) - l(u_n, T; \psi)) dt = \Delta u_n \int_0^{\overline{T}} \int_{r_1}^{r_2} r^2 (3\lambda + 2\mu) T \left(\frac{\partial \psi}{\partial r} + \frac{2\psi}{r} \right) dr \, dt, \qquad (44)$$

where T is the solution of the problem, defined by the second and third relations of the system (41).

Hence, $J'_{u_n} = \widetilde{\psi}_n$, where

$$\widetilde{\psi}_n = \int_0^{\overline{T}} \int_{r_1}^{r_2} r^2 (3\lambda + 2\mu) T \left(\frac{\partial \psi}{\partial r} + \frac{2\psi}{r} \right) dr \, dt, \quad \| J'_{u_n} \| = | \widetilde{\psi}_n |.$$

Remark 4. If $u = u(t)$, then on the basis of (44) we have $J'_{u_n} = \widetilde{\psi}_n$, where

$$\widetilde{\psi}_n = \int_{r_1}^{r_2} r^2 (3\lambda + 2\mu) T \left(\frac{\partial \psi}{\partial r} + \frac{2\psi}{r} \right) dr, \quad \| J'_{u_n} \|^2 = \int_0^{\overline{T}} (\widetilde{\psi}_n)^2 \, dt.$$

Remark 5. If $u = u(r)$, then $J'_{u_n} = \widetilde{\psi}_n$, where

$$\widetilde{\psi}_n = \int_0^{\overline{T}} r^2 (3\lambda + 2\mu) T \left(\frac{\partial \psi}{\partial r} + \frac{2\psi}{r} \right) dt, \quad \| J'_{u_n} \|^2 = \int_{r_1}^{r_2} (\widetilde{\psi}_n)^2 \, dr.$$

Remark 6. If $u = u(r, t)$, then $J'_{u_n} = \widetilde{\psi}_n$, where

$$\widetilde{\psi}_n = r^2 (3\lambda + 2\mu) T \left(\frac{\partial \psi}{\partial r} + \frac{2\psi}{r} \right), \quad \| J'_{u_n} \|^2 = \int_0^{\overline{T}} \int_{r_1}^{r_2} (\widetilde{\psi}_n)^2 \, dr \, dt.$$

4. Identification of thermostressed state on the basis of the problem of elastic equilibrium

Let the equilibrium equation have the form

$$-\left\{ (\lambda + 2\mu) \left(\frac{\partial}{\partial r} \left(r^2 \frac{\partial y}{\partial r} \right) - 2y \right) - (3\lambda + 2\mu) \alpha r^2 \frac{\partial u}{\partial r} \right\} = 0, \quad (r, t) \in \Omega_T, \qquad (45)$$

where we suppose variation of temperature $u \in \mathcal{U} = C^{2,1}(\Omega_T)$ to be unknown. On internal and external surfaces of the hollow sphere the stresses (4) are given. We assume that on external surface of the sphere displacements, given by the equality (8), are known.

We obtained the problem (45), (4), (8), which consists in determination of the function $u \in \mathcal{U}$, for which the solution $y = y(u) = y(u; r, t)$ of the boundary value problem (45), (4) holds the equality (8).

Definition 6. For every fixed $u \in \mathcal{U}$ we call as the generalized solution of the problem (45), (4) the function $y = y(u) = y(u; r, t) \in V_1$, which $\forall z_1 = z_1(r) \in V_1^0$ holds the relation

$$a(y, z_1) = l(u; z_1) \quad \forall t \in (0, \overline{T}), \qquad (46)$$

where $l(u; z_1) = \int_{r_1}^{r_2} r^2 (3\lambda + 2\mu) \alpha u \left(\frac{\partial z_1}{\partial r} + \frac{2z_1}{r} \right) dr + r_1^2 p_1 z_1(r_1) - r_2^2 p_2 z_1(r_2).$

Remark 7. On solving the problem (46), (9) we can accept $\mathscr{U} = C(\overline{\Omega}_T)$.

Instead of the problem (46), (8) we shall solve by gradient methods (13) the problem (46), (9). For every approximation u_n of solution $u \in \mathscr{U}$ of the problem (46), (9) the conjugate problem has the form (42) with the corresponding generalized (43). If we substitute in (43) the function z_1 by the difference $y(u_{n+1}) - y(u_n)$, taking into account (46) we obtain

$$\left\langle J'_{u_n}, \Delta u_n \right\rangle = \int_0^{\overline{T}} a(\psi, y(u_{n+1}) - y(u_n)) \, dt =$$

$$= \int_0^{\overline{T}} (l(u_{n+1}; \psi) - l(u_n; \psi)) \, dt = \int_0^{\overline{T}} \int_{r_1}^{r_2} r^2 (3\lambda + 2\mu) \alpha \Delta u_n \left(\frac{\partial \psi}{\partial r} + \frac{2\psi}{r} \right) dr \, dt. \tag{47}$$

Hence,

$$J'_{u_n} = \widetilde{\psi}_n, \tag{48}$$

where $\widetilde{\psi}_n = r^2 (3\lambda + 2\mu) \alpha \left(\dfrac{\partial \psi}{\partial r} + \dfrac{2\psi}{r} \right)$, $\|J'_{u_n}\|^2 = \displaystyle\int_0^{\overline{T}} \int_{r_1}^{r_2} \widetilde{\psi}_n^2 dr \, dt$.

Remark 8. If $\{\varphi_i(r)\}_{i=1}^m$ is a system of linearly independent functions, and the recoverable function u is looked for as

$$u = u_m = \sum_{i=1}^m \alpha_i(t) \varphi_i(r), \tag{48'}$$

then taking into account (47) we obtain $J'_{u_n} = \widetilde{\psi}_n$, where

$$\{\widetilde{\psi}_n\} = \{\widetilde{\psi}_n^i\}_{i=1}^m, \quad \widetilde{\psi}_n^i = \int_{r_1}^{r_2} r^2 (3\lambda + 2\mu) \alpha \varphi_i \left(\frac{\partial \psi}{\partial r} + \frac{2\psi}{r} \right) dr, \quad \|J'_{u_n}\|^2 = \sum_{i=1}^m \int_0^{\overline{T}} (\widetilde{\psi}_n^i)^2 dt. \tag{49}$$

Remark 9. If besides the point $r = r_2$ solution y of the problem (46) is also known at other ones, for example, $d_i \in \Omega$, $i = \overline{1, N}$, i.e.,

$$y(d_i, t) = f_i(t), \quad i = \overline{0, N}, \quad t \in (0, \overline{T}), \tag{50}$$

then the functional-discrepancy has the form

$$J(u) = \frac{1}{2} \sum_{i=0}^N \int_0^{\overline{T}} (y(u; d_i, t) - f_i(t))^2 dt, \tag{51}$$

where $d_0 = r_2$.

We shall solve the problem (46), (51) by means of gradient methods (13). In this case for every approximation u_n of solution $u \in \mathscr{U}$ of the problem (46), (51) the conjugate problem has the form:

$$- (\lambda + 2\mu) \left(\frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) - 2\psi \right) = 0, \quad (r, t) \in \Omega_{d_T},$$

$$[\psi]\big|_{d_i} = 0, \ \ [\sigma_r(\psi)]\big|_{d_i} = -\frac{1}{d_i^2}(y(u_n) - f_i)\big|_{d_i}, \ \ i = \overline{1, N}, \ t \in (0, \overline{T}), \tag{52}$$

$$\sigma_r(\psi)\big|_{r=r_1} = 0, \ \ \sigma_r(\psi)\big|_{r=r_2} = \frac{1}{r_2^2}(y(u_n; r_2, t) - f_0(t)), \ t \in (0, \overline{T}),$$

where the component $\sigma_r(\psi)$ is defined in section 1, $\Omega_{d_T} = \Omega_d \times (0, \overline{T})$, $\Omega_d = \bigcup_{i=0}^{N} \Omega_i$, $\Omega_i = (\tilde{d}_i, \tilde{d}_{i+1})$, $\tilde{d}_{N+1} = r_2$, $\tilde{d}_0 = r_1$, $i = \overline{1, N}$.

Definition 7. We call as the generalized solution of the boundary value problem (52) the function $\psi(r, t) \in V_{1d}$, which $\forall z_1 \in V_{1d}^0$ holds the relation

$$a(\psi, z_1) = l_\psi(y(u_n); z_1), \ t \in (0, \overline{T}), \tag{53}$$

where

$$V_{1d} = \left\{ v(r, t) : v\big|_{\Omega_i} \in W_2^1(\Omega_i), i = \overline{0, N}, [v]\big|_{d_i} = 0, i = \overline{1, N}, t \in (0, \overline{T}), \int_0^{\overline{T}} \|v\|_{W_2^1}^2 dt < \infty \right\},$$

$$\|v\|_{W_2^1}^2 = \sum_{i=0}^{N} \|v\|_{W_2^1(\Omega_i)}^2, \ \ l_\psi(y(u_n); z_1) = \sum_{i=0}^{N}(y(u_n; d_i, t) - f_i)z_1(d_i).$$

Taking into account (46), (51), (53) we can write

$$\langle J'_{u_n}, \Delta u_n \rangle = \sum_{i=0}^{N} \int_0^{\overline{T}} (y(u_n; d_i, t) - f_i)(y(u_{n+1}) - y(u_n))\big|_{d_i} dt =$$

$$= \int_0^{\overline{T}} a(\psi, y(u_{n+1}) - y(u_n)) \, dt = \int_0^{\overline{T}} \int_{r_1}^{r_2} r^2(3\lambda + 2\mu)\alpha \Delta u_n \left(\frac{\partial \psi}{\partial r} + \frac{2\psi}{r} \right) dr \, dt. \tag{54}$$

On the basis of (54) we obtain the expression (48).

5. Parametric identification of variation of body temperature

Let us consider the problem (46), (51) under the assumption that it is possible to represent the desired variation of body temperature $T = u$ as

$$u(r, t) = u_m(r, t) = \sum_{i=1}^{m} \alpha_i \varphi_i(r, t), \tag{55}$$

where $\{\varphi_i(r, t)\}_{i=1}^{m}$ is a system of linearly independent functions. Taking into account (55) on the basis of (54) we obtain

$$\langle J'_{u_n}, \Delta u_n \rangle = \sum_{i=1}^{m} \Delta \alpha_{in} \int_0^{\overline{T}} \int_{r_1}^{r_2} r^2(3\lambda + 2\mu)\alpha \varphi_i \left(\frac{\partial \psi}{\partial r} + \frac{2\psi}{r} \right) dr \, dt.$$

Hence, $J'_{u_n} = \tilde{\psi}_n$, where

$$\tilde{\psi}_n = \{\tilde{\psi}_n^i\}_{i=1}^{m}, \ \ \tilde{\psi}_n^i = \int_0^{\overline{T}} \int_{r_1}^{r_2} r^2(3\lambda + 2\mu)\alpha \varphi_i \left(\frac{\partial \psi}{\partial r} + \frac{2\psi}{r} \right) dr \, dt, \ \ \|J'_{u_n}\|^2 = \sum_{i=1}^{m}(\tilde{\psi}_n^i)^2.$$

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6. Identification by given displacements of thermostressed state of two-layered body with weakly penetrable interlayer

Let on the intervals $\Omega_1 = (r_1, \xi)$, $\Omega_2 = (\xi, r_2)$ $(0 < r_1 < \xi < r_2 < \infty)$ the equilibrium equation has the form

$$-\left\{ (\lambda + 2\mu)\left(\frac{\partial}{\partial r}\left(r^2 \frac{\partial y}{\partial r} \right) - 2y \right) - (3\lambda + 2\mu)\alpha r^2 \frac{\partial T}{\partial r} \right\} = 0, \ (r, t) \in \Omega_T, \tag{56}$$

where $\Omega_T = \Omega \times (0, \overline{T})$, $\Omega = \Omega_1 \cup \Omega_2$.

Variation of temperature T holds the relation

$$c\frac{\partial T}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 k \frac{\partial T}{\partial r} \right) + \bar{f}, \ (r, t) \in \Omega_T. \tag{57}$$

On internal and external surfaces of compound hollow sphere stresses are given

$$\sigma_r(y)\big|_{r = r_i} = -p_i, \ i = 1, 2, \ t \in (0, \overline{T}), \tag{58}$$

density of heat flow on internal surface is

$$-k\frac{\partial T}{\partial r} = u, \ r = r_1, \ t \in (0, \overline{T}), \tag{59}$$

which we suppose to be unknown, and on external surface we have boundary condition of the third kind

$$k\frac{\partial T}{\partial r} = -\overline{\alpha} T + \beta, \ r = r_2, \ t \in (0, \overline{T}). \tag{60}$$

On spherical surface of $r = \xi$ radius of contact of components of compound sphere we have conjugation conditions

$$[y] = 0, \ [\sigma_r(y)] = 0,$$

$$R_1\left\{ k\frac{\partial T}{\partial r} \right\}^- + R_2\left\{ k\frac{\partial T}{\partial r} \right\}^+ = [T], \tag{61}$$

$$\left[k\frac{\partial T}{\partial r} \right] = \omega, \ t \in (0, \overline{T}),$$

where the components $\sigma_r(y)$, $\sigma_\varphi(y)$, $\sigma_\theta(y)$ have the form (1').

For $t = 0$ we have the initial condition

$$T(r, 0) = \overline{T}_0(r), \ r \in \Omega_1 \cup \Omega_2. \tag{62}$$

We assume that on external surface of compound sphere displacement is known

$$y(r_2, t) = f_0(t), \ t \in (0, \overline{T}). \tag{63}$$

We obtained the problem (56)–(63), which consists is determination of the function $u = u(t) \in \mathcal{U} = C([0, \overline{T}])$, for which the first component y of the classical solution $Y = (y, T)$ of the initial boundary value problem (56)–(62) holds the equality (63).

Definition 8. We call as the generalized solution of the initial boundary value problem (56)–(62) the vector-function $Y = (y, T) \in V$, which $\forall z = (z_1, z_2) \in V_0$ holds the relations:

$$a(y, z_1) = l(T; z_1), \ t \in (0, \overline{T}), \tag{64}$$

$$\left(r^2 c \frac{\partial T}{\partial t}, z_2 \right) + a_1(T, z_2) = l_1(u; z_2), \ t \in (0, \overline{T}), \tag{65}$$

$$(r^2 c T, z_2)(0) = (r^2 c \overline{T}_0, z_2). \tag{66}$$

Here

$$a(y, z_1) = \int_{r_1}^{r_2} r^2 \left\{ (\lambda + 2\mu) \left(\frac{\partial y}{\partial r} \frac{\partial z_1}{\partial r} + 2 \frac{y}{r} \frac{z_1}{r} \right) + 2\lambda \left(\frac{\partial y}{\partial r} \frac{z_1}{r} + \frac{y}{r} \frac{\partial z_1}{\partial r} + \frac{y}{r} \frac{z_1}{r} \right) \right\} dr,$$

$$a_1(T, z_2) = \int_{r_1}^{r_2} r^2 k \frac{\partial T}{\partial r} \frac{\partial z_2}{\partial r} dr + \xi^2 \frac{[T][z_2]}{R_1 + R_2} + \overline{\alpha} r_2^2 T z_2 \bigg|_{r=r_2},$$

$$l(T; z_1) = \int_{r_1}^{r_2} r^2 (3\lambda + 2\mu) \alpha T \left(\frac{\partial z_1}{\partial r} + \frac{2 z_1}{r} \right) dr + r_1^2 p_1 z_1(r_1) - r_2^2 p_2 z_1(r_2),$$

$$l_1(u; z_2) = \int_{r_1}^{r_2} r^2 \overline{f} z_2 dr + \xi^2 \frac{R_2 \omega}{R_1 + R_2} [z_2] - \xi^2 \omega z_2^+ + r_1^2 u z_2(r_1) + \beta r_2^2 z_2(r_2),$$

$$V = \left\{ v = (v_1(r, t), v_2(r, t)) : v_i \big|_{\Omega_j} \in W_2^1(\Omega_j), \frac{\partial v_2}{\partial t} \bigg|_{\Omega_i} \in L^2(0, \overline{T}; L_2(\Omega_i)), i, j = 1, 2; \right.$$

$$\left. [v_1] \big|_{r=\xi} = 0, \forall t \in [0, \overline{T}], \int_0^{\overline{T}} \sum_{j=1}^2 \| v_i \|_{W_2^1(\Omega_j)}^2 dt < \infty, i = 1, 2 \right\},$$

$$V_0 = \{ v = (v_1(r), v_2(r)) : v_i \big|_{\Omega_j} \in W_2^1(\Omega_j), i, j = 1, 2, [v_1] \big|_{r=\xi} = 0 \}.$$

The functional-discrepancy has the form (9). The expressions of (17)–(19) type are valid. Foe every approximation u_n of solution $u \in \mathcal{U} = C([0, \overline{T}])$ of the problem (64)–(66), (9) the conjugate problem has the form:

$$-(\lambda + 2\mu) \left(\frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) - 2\psi \right) = 0, \ (r, t) \in \Omega_T,$$

$$\sigma_r(\psi) \big|_{r=r_1} = 0, \ \sigma_r(\psi) \big|_{r=r_2} = \frac{1}{r_2^2} (y(u_n) - f_0) \big|_{r=r_2}, \ t \in (0, \overline{T}),$$

$$-r^2 c \frac{\partial p}{\partial t} - \frac{\partial}{\partial r} \left(r^2 k \frac{\partial p}{\partial r} \right) - r^2 (3\lambda + 2\mu) \alpha \left(\frac{\partial \psi}{\partial r} + \frac{2\psi}{r} \right) = 0, \ (r, t) \in \Omega_T,$$

$$-k \frac{\partial p}{\partial r} \bigg|_{r=r_1} = 0, \ k \frac{\partial p}{\partial r} \bigg|_{r=r_2} = -\overline{\alpha} p(r_2, t), \ t \in (0, \overline{T}), \tag{67}$$

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$$[\psi]\big|_{r=\xi} = 0, \quad [\sigma_r(\psi)]\big|_{r=\xi} = 0,$$

$$\left[k\frac{\partial p}{\partial r}\right]\bigg|_{r=\xi} = 0, \quad \left\{k\frac{\partial p}{\partial r}\right\}^{\pm} = \frac{[p]}{R_1 + R_2},$$

$$p\big|_{t=\overline{T}} = 0, \quad r \in \overline{\Omega},$$

where the component $\sigma_r(\psi)$ is defined in section 1.

Definition 9. We call as the generalized solution of the initial boundary value problem (67) the vector-function $Y^* = (\psi, p) \in V$, which $\forall z = (z_1, z_2) \in V_0$ holds the relations

$$a(\psi, z_1) = (y(u_n; r_2, t) - f_0)z_1(r_2), \quad t \in (0, \overline{T}), \tag{68}$$

$$-\left(r^2 c\frac{\partial p}{\partial t}, z_2\right) + a_1(p, z_2) - \int_{\Omega} r^2(3\lambda + 2\mu) z_2 \alpha\left(\frac{\partial \psi}{\partial r} + \frac{2\psi}{r}\right)dr = 0, \quad t \in (0, \overline{T}), \tag{69}$$

$$(r^2 cp, z_2)(\overline{T}) = 0. \tag{70}$$

If we select in the relation (68) instead of the function z_1 the difference $y(u_{n+1}) - y(u_n)$, and the difference $T(u_{n+1}) - T(u_n)$ instead of z_2 in the relations (69), (70), taking into account (64)–(66) we obtain

$$\int_0^{\overline{T}} (y(u_n; r_2, t) - f_0)(y(u_{n+1}; r_2, t) - y(u_n; r_2, t))\,dt = \int_0^{\overline{T}} a(y(u_{n+1}) - y(u_n), \psi)\,dt -$$

$$- \int_0^{\overline{T}}\int_{\Omega} r^2(3\lambda + 2\mu)\,\alpha\,(T(u_{n+1}) - T(u_n))\left(\frac{\partial \psi}{\partial r} + \frac{2\psi}{r}\right)dr\,dt + \int_0^{\overline{T}} a_1(T(u_{n+1}) - T(u_n), p)\,dt =$$

$$= \int_0^{\overline{T}} (l_1(u_{n+1}; p) - l_1(u_n; p))\,dt = \int_0^{\overline{T}} \Delta u_n r_1^2 p(r_1, t)\,dt. \tag{71}$$

Taking into account (19) on the basis of (71) we have

$$\left\langle J'_{u_n}, \Delta u_n \right\rangle = \int_0^{\overline{T}} \Delta u_n r_1^2 p(r_1, t)\,dt. \tag{72}$$

Hence, $J'_{u_n} = \widetilde{\psi}_n$, where

$$\widetilde{\psi}_n = r_1^2 p(r_1, t), \quad \left\|J'_{u_n}\right\|^2 = \int_0^{\overline{T}} \widetilde{\psi}_n^2\,dt. \tag{73}$$

The presence of the gradient J'_{u_n} makes it possible to use gradient methods (13) for determination of the $(n+1)$-th approximation u_{n+1} of solution $u \in \mathcal{U}$ of the problem (64)–(66), (9).

Remark 10. If representation (26) takes place, then the expressions (26′) are valid.

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7. Simultaneous identification of density of heat flow and thermal resistance

Let on the domain $\Omega_T = \Omega \times (0, \overline{T})$ $(\Omega = \Omega_1 \cup \Omega_2)$ the equation of elastic equilibrium be given

$$-\left\{ (\lambda + 2\mu) \left(\frac{\partial}{\partial r} \left(r^2 \frac{\partial y}{\partial r} \right) - 2y \right) - (3\lambda + 2\mu) \alpha r^2 \frac{\partial T}{\partial r} \right\} = 0 \tag{74}$$

and the diffusion equation

$$c \frac{\partial T}{\partial t} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 k \frac{\partial T}{\partial r} \right) + \bar{f}. \tag{75}$$

At the ends of segment $[r_1, r_2]$ boundary conditions (58), (60) and the constraint are given

$$-k \frac{\partial T}{\partial r} = u_1, \ \ r = r_1, \ \ t \in (0, \overline{T}). \tag{76}$$

At the point $r = \xi$ conjugation conditions have the form

$$[y] = 0, \ \ [\sigma_r(y)] = 0,$$

$$\left[k \frac{\partial T}{\partial r} \right] = 0, \ \ \left\{ k \frac{\partial T}{\partial r} \right\}^{\pm} = u_2[T], \ \ t \in (0, \overline{T}). \tag{77}$$

For $t = 0$ we have the initial condition

$$T(r, 0) = \overline{T}_0(r), \ \ r \in \Omega_1 \cup \Omega_2. \tag{78}$$

Let us imagine that on external surface of the sphere at the point $r = \xi$ and at certain internal points $d_i \in \Omega, \ \ i = \overline{2, N}$, displacements are known and are given by the equalities

$$y(d_i, t) = f_i(t), \ \ i = \overline{0, N}, \ \ d_0 = r_2, \ \ d_1 = \xi, \ \ t \in (0, \overline{T}). \tag{79}$$

So, we obtained the problem (74)–(79), (58), (60), which consists in determination of a vector $u = (u_1, u_2) \in \mathcal{U} = C([0, \overline{T}]) \times C_+([0, \overline{T}])$, for which the first component y of the classical solution $Y = (y, T)$ of the initial boundary value problem (74)–(78), (58), (60) holds the equalities (79).

Definition 10. We call as the generalized solution of the initial boundary value problem (74)–(78), (58)–(60) the vector-function $Y = (y, T) \in V$, which $\forall z = (z_1, z_2) \in V_0$ holds the relations

$$a(y, z_1) = l(T; z_1), \ \ t \in (0, \overline{T}), \tag{80}$$

$$\left(r^2 c \frac{\partial T}{\partial t}, z_2 \right) + a_1(u; T, z_2) = l_1(u; z_2), \ \ t \in (0, \overline{T}), \tag{81}$$

$$(r^2 cT, z_2)(0) = (r^2 c \overline{T}_0, z_2), \tag{82}$$

where the sets V, V_0, the forms $a(\cdot, \cdot), l(\cdot; \cdot)$ are defined in section 6,

$$a_1(u; T, z_2) = \int\limits_{r_1}^{r_2} r^2 k \frac{\partial T}{\partial r} \frac{\partial z_2}{\partial r} dr + \xi^2 u_2[T][z_2] + \overline{\alpha} r_2^2 T(r_2, t) z_2(r_2),$$

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$$l_1(u; z_2) = \int_{r_1}^{r_2} r^2 \bar{f} z_2 dr + r_1^2 u_1 z_2(r_1) + \beta r_2^2 z_2(r_2).$$

Functional-discrepancy has the form

$$J(u) = \frac{1}{2} \sum_{i=0}^{N} \int_{0}^{\bar{T}} (y(u; d_i, t) - f_i(t))^2 dt. \tag{83}$$

We shall solve the obtained problem (80)–(83) by means of gradient methods (13). For every approximation u_n of solution $u \in \mathcal{U}$ of the problem (80)–(83) for $\forall u, \ v \in \mathcal{U}$ we introduce denotations

$$\begin{aligned}
\pi(u, v) &= (\bar{y}(u) - \bar{y}(u_n), \bar{y}(v) - \bar{y}(u_n))_{L_2}, \\
L(v) &= (f - \bar{y}(u_n), \bar{y}(v) - \bar{y}(u_n))_{L_2},
\end{aligned} \tag{84}$$

where $\forall v \in \mathcal{U} \ \ \bar{y}(v) = Av$, $Av = \{y(v; d_i, t)\}_{i=0}^{N}$, $y(v; r, t)$ is the first component of solution $Y = (y, T)$ of the problem (80)–(82) for $u = v$, $(\bar{\varphi}, \bar{\psi})_{L_2} = \sum_{i=0}^{N} \int_{0}^{\bar{T}} \varphi_i \psi_i \, dt$, $\bar{\varphi} = \{\varphi_i(t)\}_{i=0}^{N}$, $\bar{\psi} = \{\psi_i(t)\}_{i=0}^{N}$, $f = \{f_i\}_{i=0}^{N}$

The following expression takes place

$$2J(v) = \pi(v, v) - 2L(v) + \left\| f - \bar{y}(u_n) \right\|_{L_2}^2. \tag{85}$$

On the basis of the problem (80)–(83) for every approximation u_n by omitting terms of the second order of smallness we determine the function $\tilde{Y} = (\tilde{y}, \tilde{T}) \in V$ as solution of the problem

$$a(\tilde{y}, z_1) = l(\tilde{T}; z_1), \ \ t \in (0, \bar{T}),$$

$$\left(r^2 c \frac{\partial \tilde{T}}{\partial t}, z_2 \right) + a_1(u_n; \tilde{T}, z_2) =$$

$$= l_1(u_n; z_2) - \xi^2 \Delta u_{2n}[T(u_n)][z_2] + \Delta u_{1n} r_1^2 z_2(r_1), \ \ t \in (0, \bar{T}), \tag{86}$$

$$(r^2 c \tilde{T}, z_2)(0) = (r^2 c \bar{T}_0, z_2) \ \forall z = (z_1, z_2) \in V_0.$$

We have

$$\bar{y}(u_n + \lambda \Delta u_n) - \bar{y}(u_n) \approx \lambda(\bar{\tilde{y}}(u_{n+1}) - \bar{y}(u_n)), \tag{87}$$

where $\bar{\tilde{y}}(u_{n+1}) = \{\tilde{y}(u_{n+1}; d_i, t)\}_{i=0}^{N}$.

Taking into account (87), (84), (85) we can write

$$\left\langle J'_{u_n}, \Delta u_n \right\rangle = \lim_{\lambda \to 0} \frac{J(u_n + \lambda \Delta u_n) - J(u_n)}{\lambda} \approx (\bar{y}(u_n) - f, \bar{\tilde{y}}(u_{n+1}) - \bar{y}(u_n))_{L_2}.$$

For every approximation u_n of solution $u \in \mathcal{U}$ of the problem (80)–(83) we introduce the following conjugate problem:

$$-(\lambda + 2\mu)\left(\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r} \right) - 2\psi \right) = 0, \ \ (r, t) \in \Omega_{d_T},$$

$$-r^2 c \frac{\partial p}{\partial t} - \frac{\partial}{\partial r}\left(r^2 k \frac{\partial p}{\partial r}\right) - r^2 (3\lambda + 2\mu)\,\alpha\left(\frac{\partial \psi}{\partial r} + \frac{2\psi}{r}\right) = 0, \ (r,t) \in \Omega_{d_T},$$

$$\sigma_r(\psi)\big|_{r=r_1} = 0, \ \sigma_r(\psi)\big|_{r=r_2} = \frac{1}{r_2^2}(y(u_n) - f_0)\big|_{r=r_2},$$

$$-k\frac{\partial p}{\partial r}\bigg|_{r=r_1} = 0, \ k\frac{\partial p}{\partial r}\bigg|_{r=r_2} = -\overline{\alpha} p(r_2, t), \ t \in (0, \overline{T}),$$

$$[\psi]\big|_{d_i} = 0, \ [\sigma_r(\psi)]\big|_{d_i} = -\frac{1}{d_i^2}(y(u_n) - f_i)\big|_{r=d_i}, \ i = \overline{1, N}, \ t \in (0, \overline{T}), \qquad (88)$$

$$[p]\big|_{d_i} = 0 \ \left[k\frac{\partial p}{\partial r}\right]\bigg|_{d_i} = 0, \ i = \overline{2, N}, \ t \in (0, \overline{T}),$$

$$\left[k\frac{\partial p}{\partial r}\right] = 0, \ \left\{k\frac{\partial p}{\partial r}\right\}^{\pm} = u_{2n}[p], \ r = \xi, \ t \in (0, \overline{T}),$$

$$p\big|_{t=\overline{T}} = 0, \ r \in \overline{\Omega},$$

where $\Omega_{d_T} = \Omega_d \times (0, \overline{T})$, $\Omega_d = \Omega \setminus \gamma_d$, $\gamma_d = \bigcup\limits_{i=2}^{N} d_i$.

Definition 11. We call as the generalized solution of the initial boundary value problem (88) the vector-function $Y^* = (\psi, p) \in V_d$, which $\forall z = (z_1, z_2) \in V_d^0$ holds the relations

$$a(\psi, z_1) = \sum_{i=0}^{N}(y(u_n; d_i, t) - f_i(t)) z_1(d_i), \ t \in (0, \overline{T}), \qquad (89)$$

$$-\left(r^2 c \frac{\partial p}{\partial t}, z_2\right) + a_1(u_n; p, z_2) - \int\limits_{\Omega} r^2 (3\lambda + 2\mu) z_2 \alpha \left(\frac{\partial \psi}{\partial r} + \frac{2\psi}{r}\right) dr = 0, \ t \in (0, \overline{T}), \qquad (90)$$

$$(r^2 c p, z_2)(\overline{T}) = 0, \qquad (91)$$

where

$$V_d = \left\{ v = (v_1(r,t), v_2(r,t)) : v_i\big|_{\Omega_j} \in W_2^1(\Omega_j), i = 1, 2; j = \overline{1, N+1}, \right.$$

$$\frac{\partial v_2}{\partial t}\bigg|_{\Omega_j} \in L^2(0, \overline{T}; L_2(\Omega_j)), j = \overline{1, N+1}, [v_1]\big|_{d_i} = 0, i = \overline{1, N}; [v_2]\big|_{d_i} = 0, i = \overline{2, N};$$

$$\left. \int\limits_{0}^{\overline{T}} \sum_{j=1}^{N+1} \|v_i\|_{W_2^1(\Omega_j)}^2 dt < \infty, i = 1, 2 \right\},$$

$$V_d^0 = \left\{ v = (v_1(r), v_2(r)) : v_i\big|_{\Omega_j} \in W_2^1(\Omega_j), i = 1, 2; [v_1]\big|_{d_i} = 0, i = \overline{1, N}, [v_2]\big|_{d_i} = 0, i = \overline{2, N} \right\}.$$

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If we select in the relation (89) instead of the function z_1 the difference $\tilde{y}(u_{n+1}) - y(u_n)$ and the difference $\tilde{T}(u_{n+1}) - T(u_n)$ instead of z_2 in (90), (91), taking into account (86) we assume

$$\left\langle J'_{u_n}, \Delta u_n \right\rangle \approx (\bar{y}(u_n) - f, \bar{\tilde{y}}(u_{n+1}) - \bar{y}(u_n))_{L_2} = \int\limits_0^{\overline{T}} a(\psi, \tilde{y}(u_{n+1}) - y(u_n))\, dt -$$

$$- \int\limits_0^{\overline{T}} \int\limits_\Omega r^2 (3\lambda + 2\mu)(\tilde{T}(u_{n+1}) - T(u_n))\alpha \left(\frac{\partial \psi}{\partial r} + \frac{2\psi}{r} \right) dr\, dt +$$

$$+ \int\limits_0^{\overline{T}} a_1(u_n; \tilde{T}(u_{n+1}) - T(u_n), p)\, dt = \int\limits_0^{\overline{T}} \Delta u_{1n} r_1^2 p(r_1, t)\, dt - \int\limits_0^{\overline{T}} \xi^2 \Delta u_{2n}[T(u_n)][p]\, dt. \qquad (92)$$

Therefore,

$$J'_{u_n} \approx \tilde{\psi}_n, \qquad (93)$$

where $\tilde{\psi}_n = \{\tilde{\psi}_n^i\}_{i=1}^2$, $\tilde{\psi}_n^1 = r_1^2 p(r_1, t)$, $\tilde{\psi}_n^2 = -\xi^2[T(u_n)][p]$, $\left\| J'_{u_n} \right\|^2 \approx \int\limits_0^{\overline{T}} \sum\limits_{i=1}^2 (\tilde{\psi}_n^i)^2\, dt$.

Remark 11. On the basis of the expression (92) we can easily obtain approximation of the gradient J'_{u_n} (93) in the case of parametric representation of one or two parameters u_1, u_2, simultaneously, i.e., representing them similar to (55).

Remark 12. If u_1, $u_2 = \mathrm{const}$, then on the basis of (92) we obtain $J'_{u_n} \approx \tilde{\psi}_n$, where

$$\tilde{\psi}_n = \{\tilde{\psi}_n^i\}_{i=1}^2, \quad \tilde{\psi}_n^1 = \int\limits_0^{\overline{T}} r_1^2 p(r_1, t)\, dt, \quad \tilde{\psi}_n^2 = -\int\limits_0^{\overline{T}} \xi^2[T(u_n)][p]\, dt, \quad \left\| J'_{u_n} \right\|^2 \approx \sum\limits_{i=1}^2 (\tilde{\psi}_n)^2.$$

8. Identification of coefficients of heat conductivity of components of compound hollow sphere

Let on the domain Ω_T $(\Omega = \Omega_1 \cup \Omega_2)$ the equation of elastic equilibrium be defined

$$-\left\{ (\lambda + 2\mu)\left(\frac{\partial}{\partial r}\left(r^2 \frac{\partial y}{\partial r} \right) - 2y \right) - (3\lambda + 2\mu)\alpha r^2 \frac{\partial T}{\partial r} \right\} = 0, \ (r, t) \in \Omega_T. \qquad (94)$$

Variation of temperature T holds the equation

$$c\frac{\partial T}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u \frac{\partial T}{\partial r} \right) + \bar{f}, \ (r, t) \in \Omega_T, \qquad (95)$$

where $u = (u_1, u_2) \in \mathscr{U} = (0, +\infty) \times (0, +\infty)$, $u_i = u\big|_{\overline{\Omega}_i}$, $i = 1, 2$.

At the ends of the segment $[r_1, r_2]$ boundary conditions are given

$$\sigma_r(y)\big|_{r=r_i} = -p_i, \ i = 1, 2,$$

$$-u_1 \frac{\partial T}{\partial r}\bigg|_{r=r_1} = \beta_1, \ u_2 \frac{\partial T}{\partial r}\bigg|_{r=r_2} = -\overline{\alpha} T\big|_{r=r_2} + \beta_2, \ t \in (0, \overline{T}). \qquad (96)$$

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At the point $r = \xi$ $\forall t \in (0, \overline{T})$ conjugation conditions have the form

$$[y] = 0, \ [\sigma_r(y)] = 0,$$

$$\left[u \frac{\partial T}{\partial r} \right] = 0, \ \left\{ u \frac{\partial T}{\partial r} \right\}^{\pm} = \overline{r}[T]. \tag{97}$$

For $t = 0$ we have the initial condition

$$T\big|_{t=0} = \overline{T}_0, \ r \in \overline{\Omega}_1 \cup \overline{\Omega}_2. \tag{98}$$

We assume that on external surface of the sphere at the point $r = \xi$ and at some internal points $d_i \in \Omega$, $i = \overline{2, N}$ $\forall t \in (0, \overline{T})$ displacements, given by the equalities (79), are known.

We obtained the problem (94)–(98), (79), which consists in determination of the vector $u = (u_1, u_2) \in \mathcal{U}$, for which the first component y of the classical solution $Y = (y, T)$ of the initial boundary value problem (94)–(98) holds the equalities (79). The functional-discrepancy has the form (83).

For every fixed $u \in \mathcal{U}$ instead of the classical solution of the initial boundary value problem (94)–(98) we shall use its generalized solution.

Definition 12. For every fixed $u \in \mathcal{U}$ we call as the generalized solution of the initial boundary value problem (94)–(98) the vector function $Y = (y, T) \in V$, which $\forall z = (z_1, z_2) \in V_0$ holds the relations

$$a(y, z_1) = l(T; z_1), \ t \in (0, \overline{T}), \tag{99}$$

$$\left(r^2 c \frac{\partial T}{\partial t}, z_2 \right) + a_1(u; T, z_2) = l_1(z_2), \ t \in (0, \overline{T}), \tag{100}$$

$$(r^2 c T, z_2)(0) = (r^2 c \overline{T}_0, z_2), \tag{101}$$

where the sets V, V_0, the forms $a(\cdot, \cdot)$, $l(\cdot; \cdot)$ are defined in section 6,

$$a_1(u; T, z_2) = \int\limits_{r_1}^{r_2} r^2 u \frac{\partial T}{\partial r} \frac{\partial z_2}{\partial r} dr + \xi^2 \overline{r}[T][z_2] + \overline{\alpha} r_2^2 T(r_2, t) z_2(r_2),$$

$$l_1(z_2) = \int\limits_{r_1}^{r_2} r^2 \overline{f} z_2 dr + r_1^2 \beta_1 z_2(r_1) + \beta_2 \, r_2^2 z_2(r_2).$$

For every increment $\theta = \Delta Y$ of the solution $Y(u)$, which corresponds to increment Δu of the element $u \in \mathcal{U}$, on the basis of the boundary value problem (94)–(98) we obtain the following initial boundary value problem:

$$-\left\{ (\lambda + 2\mu) \left(\frac{\partial}{\partial r} \left(r^2 \frac{d\theta_1}{dr} \right) - 2\theta_1 \right) - (3\lambda + 2\mu) \alpha r^2 \frac{\partial \theta_2}{\partial r} \right\} = 0, \ (r, t) \in \Omega_T,$$

$$c \frac{\partial \theta_2}{\partial t} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 u \frac{\partial \theta_2}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \Delta u \frac{\partial T(u)}{\partial r} \right), \ (r, t) \in \Omega_T,$$

$$\sigma_r(\theta)\big|_{r=r_i} = 0, \ i = 1, 2, \ t \in (0, \overline{T}),$$

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$$-u_1 \frac{\partial \theta_2}{\partial r} = \Delta u_1 \frac{\partial T(u)}{\partial r}, \ \ r = r_1, \ t \in (0, \overline{T}), \tag{102}$$

$$u_2 \frac{\partial \theta_2}{\partial r} = -\overline{\alpha} \theta_2 - \Delta u_2 \frac{\partial T(u)}{\partial r}, \ \ r = r_2, \ t \in (0, \overline{T}),$$

$$[\theta_1] = [\sigma_r(\theta)] = 0, \ \ r = \xi, \ t \in (0, \overline{T}),$$

$$\left[u \frac{\partial \theta_2}{\partial r} \right] = -\left[\Delta u \frac{\partial T(u)}{\partial r} \right], \ \left\{ u \frac{\partial \theta_2}{\partial r} \right\}^{\pm} = \overline{r}[\theta_2] - \left\{ \Delta u \frac{\partial T(u)}{\partial r} \right\}^{\pm}, \ \ r = \xi, \ t \in (0, \overline{T}),$$

$$\theta_2 \big|_{t=0} = 0, \ \ r \in \overline{\Omega},$$

where $u^+ = u_2$, $u^- = u_1$, $\Delta u^+ = \Delta u_2$, $\Delta u^- = \Delta u_1$, $\theta = (\theta_1, \theta_2)$, $\theta_1 = \Delta y$, $\theta_2 = \Delta T$.

Definition 13. We call as the generalized solution of the initial boundary value problem (102) the vector-function $\theta \in V$, which $\forall z = (z_1, z_2) \in V_0$ holds the system of relations

$$a(\theta_1, z_1) = l_\theta^1 (\Delta u; z_1), \ \ t \in (0, \overline{T}),$$

$$\left(r^2 c \frac{\partial \theta_2}{\partial t}, z_2 \right) + a_1(u; \theta_2, z_2) = l_{1\theta}^1 (\Delta u, T(u); z_2), \ \ t \in (0, \overline{T}), \tag{103}$$

$$(r^2 c \theta_2, z_2)(0) = 0,$$

where

$$l_\theta^1 (\Delta u; z_1) = \int_{r_1}^{r_2} r^2 (3\lambda + 2\mu) \alpha \theta_2 \left(\frac{\partial z_1}{\partial r} + \frac{2 z_1}{r} \right) dr,$$

$$l_{1\theta}^1 (\Delta u, T(u); z_2) = \sum_{i=1}^{2} \int_{\Omega_i} \frac{\partial}{\partial r} \left(\Delta u_i r^2 \frac{\partial T(u)}{\partial r} \right) z_2 dr + \Delta u_1 r_1^2 \frac{\partial T(u)}{\partial r} \bigg|_{r=r_1} z_2 (r_1) -$$

$$- r_2^2 \Delta u_2 \frac{\partial T(u)}{\partial r} \bigg|_{r=r_2} z_2 (r_2) + \Delta u_2 \xi^2 \left\{ \frac{\partial T(u)}{\partial r} \right\}^+ z_2^+ - \Delta u_1 \xi^2 \left\{ \frac{\partial T(u)}{\partial r} \right\}^- z_2^-.$$

The expressions (84), (85), (87) take place, where $\widetilde{y}(u_{n+1}) = y(u_n) + \theta_1$.

For every approximation u_n of solution $u \in \mathcal{U}$ of the problem (99)–(101), (83) we write the conjugate problem in the following way:

$$-(\lambda + 2\mu) \left(\frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) - 2\psi \right) = 0, \ \ (r, t) \in \Omega_{d_T},$$

$$-r^2 c \frac{\partial p}{\partial t} - \frac{\partial}{\partial r} \left(r^2 u_n \frac{\partial p}{\partial r} \right) - r^2 (3\lambda + 2\mu) \alpha \left(\frac{\partial \psi}{\partial r} + \frac{2\psi}{r} \right) = 0, \ \ (r, t) \in \Omega_{d_T},$$

$$-u_{1n} \frac{\partial p}{\partial r} \bigg|_{r=r_1} = 0, \ \ u_{2n} \frac{\partial p}{\partial r} \bigg|_{r=r_2} = -\overline{\alpha} p \big|_{r=r_2}, \ \ t \in (0, \overline{T}),$$

$$[\psi]\big|_{r=d_i} = 0, \ [\sigma_r(\psi)]\big|_{r=d_i} = -\frac{1}{d_i^2}(y(u_n;d_i,t)-f_i), \ i=\overline{1,N},$$

$$(104)$$

$$[p]\big|_{d_i} = 0, \ \left[u_n\frac{\partial p}{\partial r}\right]\bigg|_{d_i} = 0, \ i=\overline{2,N},$$

$$\sigma_r(\psi)\big|_{r=r_1} = 0, \ \sigma_r(\psi)\big|_{r=r_2} = \frac{1}{r_2^2}(y(u_n;r_2,t)-f_0),$$

$$\left[u_n\frac{\partial p}{\partial r}\right] = 0, \ \left\{u_n\frac{\partial p}{\partial r}\right\}^{\pm} = \bar{r}[p], \ r=\xi, \ t\in(0,\overline{T}),$$

$$p\big|_{t=\overline{T}} = 0, \ r\in\overline{\Omega},$$

where $\sigma_r(\psi) = (\lambda+2\mu)\dfrac{\partial\psi}{\partial r} + 2\lambda\dfrac{\psi}{r}$, $[\varphi]\big|_{d_i} = [\varphi]\big|_{r=d_i}$.

Definition 14. We call as the generalized solution of the initial boundary value problem (104) the vector-function $Y^* = (\psi, p)\in V_d$, which $\forall z = (z_1, z_2)\in V_d^0$ holds the relations

$$a(\psi, z_1) = \sum_{i=0}^{N}(y(u_n;d_i,t)-f_i)z_1(d_i), \ t\in(0,\overline{T}),$$

$$(105)$$

$$-\left(r^2c\frac{\partial p}{\partial t}, z_2\right) + a_1(u;p,z_2) - \int_{\Omega} r^2(3\lambda+2\mu)z_2\alpha\left(\frac{\partial\psi}{\partial r}+\frac{2\psi}{r}\right)dr = 0, \ t\in(0,\overline{T}),$$

$$(106)$$

$$(r^2cp, z_2)(\overline{T}) = 0,$$

$$(107)$$

where the sets V_d, V_d^0 are defined in section 7, $u=u_n$.

If we select in the relation (105) instead of the function z_1 the difference $\tilde{y}(u_{n+1})-y(u_n)$, and the difference $\tilde{T}(u_{n+1})-T(u_n)$ instead of z_2 in (106), (107), taking into account (103) we get

$$\left\langle J'_{u_n}, \Delta u_n \right\rangle \approx (\bar{y}(u_n)-f, \ \tilde{\bar{y}}(u_{n+1})-\bar{y}(u_n))_{L_2} = \int_0^{\overline{T}} a(\psi, \tilde{y}(u_{n+1})-\bar{y}(u_n))\,dt -$$

$$-\int_0^{\overline{T}}\int_{\Omega} r^2(3\lambda+2\mu)(\tilde{T}(u_{n+1})-T(u_n))\alpha\left(\frac{\partial\psi}{\partial r}+\frac{2\psi}{r}\right)dr\,dt +$$

$$+\int_0^{\overline{T}} a_1(u_n; \tilde{T}(u_{n+1})-T(u_n), p)\,dt = \int_0^{\overline{T}} l_{1\theta}^1(\Delta u_n, T(u_n); p)\,dt.$$

$$(108)$$

Therefore,

$$J'_{u_n} \approx \tilde{\psi}_n,$$

where

$$\tilde{\psi}_n = \{\tilde{\psi}_n^i\}_{i=1}^2,$$

$$\tilde{\psi}_n^1 = \int_0^{\overline{T}}\int_{r_1}^{\xi}\frac{\partial}{\partial r}\left(r^2\frac{\partial T(u_n)}{\partial r}\right)p\,dr\,dt + \int_0^{\overline{T}} r_1^2\frac{\partial T(u_n)}{\partial r}p\bigg|_{r=r_1}dt - \int_0^{\overline{T}}\xi^2\left\{\frac{\partial T(u_n)}{\partial r}\right\}^{-}p^{-}\,dt,$$

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$$\tilde{\psi}_n^2 = \int\limits_0^{\overline{T}}\int\limits_\xi^{r_2} \frac{\partial}{\partial r}\left(r^2\frac{\partial T(u_n)}{\partial r}\right)p\,dr\,dt - \int\limits_0^{\overline{T}} r_2^2\frac{\partial T(u_n)}{\partial r}\,p\bigg|_{r=r_2}dt + \int\limits_0^{\overline{T}}\xi^2\left\{\frac{\partial T(u_n)}{\partial r}\right\}^+ p^+\,dt,$$

$$\left\|J'_{u_n}\right\|^2 \approx \sum_{i=1}^{2}(\tilde{\psi}_n^i)^2.$$

Remark 13. On the basis of the expression (108) we can obtain representation of approximation $\tilde{\psi}_n$ of gradient J'_{u_n} under other assumptions about the set $\mathcal{U}$, for example, parametric representations changeable from variables r, t and other.

9. Identification of thermostressed state by given displacements (inhomogeneous mixed conjugation conditions)

Let on the domain Ω_T $(\Omega = \Omega_1 \cup \Omega_2)$ the equation of elastic equilibrium is defined

$$-\left\{(\lambda + 2\mu)\left(\frac{\partial}{\partial r}\left(r^2\frac{\partial y}{\partial r}\right) - 2y\right) - (3\lambda + 2\mu)\alpha r^2\frac{\partial T}{\partial r}\right\} = 0, \ (r,t)\in\Omega_T. \tag{109}$$

Variation of temperature T holds the equation

$$c\frac{\partial T}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 k\frac{\partial T}{\partial r}\right) + \bar{f}, \ (r,t)\in\Omega_T. \tag{110}$$

At the ends of segment $[r_1, r_2]$ $\forall t \in (0, \overline{T})$ boundary conditions are given

$$\sigma_r(y)\big|_{r=r_i} = -p_i, \ i = 1, 2,$$

$$-k\frac{\partial T}{\partial r}\bigg|_{r=r_1} = u, \ k\frac{\partial T}{\partial r}\bigg|_{r=r_2} = -\overline{\alpha}T\big|_{r=r_2} + \beta. \tag{111}$$

At the point $r = \xi$ $\forall t \in (0, \overline{T})$ conjugation conditions for disjointing pressure [11, 12] and compound weakly penetrable interlayer have the form

$$[y] = \delta, \ \{\sigma_r(y)\}^- = -\overline{p}, \ \{\sigma_r(y)\}^+ = \overline{p},$$

$$R_1\left\{k\frac{\partial T}{\partial r}\right\}^- + R_2\left\{k\frac{\partial T}{\partial r}\right\}^+ = [T], \tag{112}$$

$$\left[k\frac{\partial T}{\partial r}\right] = \omega,$$

where R_1, $R_2 = \text{const}$, $R_1 + R_2 > 0$, $\delta = \delta(t)\in C([0,\overline{T}])$, $\overline{p}$ is the value of disjointing pressure.

For $t = 0$ the initial condition (98) is given. We assume that at the point $r = r_2$ displacement is known

$$y(r_2, t) = f_0(t), \ t \in (0, \overline{T}). \tag{113}$$

We obtained the problem (109)–(113), (98), which consists in determination of a real function $u = u(t)\in\mathcal{U} = C([0,\overline{T}])$, for which the first component y of the classical solution $Y = (y, T)$ of the initial boundary value problem (109)–(112), (98) holds the equality (113). Instead of the classical solution of the boundary value problem (109)–(112), (98) we shall use its generalized solution.

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Definition 15. For every fixed $u \in \mathcal{U}$ we call as the generalized solution of the initial boundary value problem (109)–(112), (98) the vector-function $Y = (y, T) \in V$, which $\forall z = (z_1, z_2) \in V_0$ holds the system of relations

$$a(y, z_1) = l(T; z_1), \ t \in (0, \overline{T}),$$

$$\left(r^2 c \frac{\partial T}{\partial t}, z_2 \right) + a_1(T, z_2) = l_1(u; z_2), \ t \in (0, \overline{T}), \tag{114}$$

$$(r^2 c T, z_2)(0) = (r^2 c \overline{T}_0, z_2),$$

where

$$V = \left\{ v = (v_1, v_2) \in \overline{V} : [v_1]\big|_{r=\xi} = \delta, \forall t \in (0, \overline{T}), \frac{\partial v_2}{\partial t}\bigg|_{\Omega_i} \in L^2(0, \overline{T}; L_2(\Omega_i)), i = 1, 2 \right\},$$

$$V_0 = \{ v \in \overline{V}_0 : [v_1]\big|_{r=\xi} = 0 \},$$

$$\overline{V} = \left\{ v = (v_1(r, t), v_2(r, t)) : v_i\big|_{\Omega_j} \in W_2^1(\Omega_j), \int_0^{\overline{T}} \|v_i\|_{W_2^1(\Omega_j)}^2 \, dt < \infty, i, j = 1, 2; t \in (0, \overline{T}) \right\},$$

$$\overline{V}_0 = \{ v = (v_1(r), v_2(r)) : v_i\big|_{\Omega_j} \in W_2^1(\Omega_j), i, j = 1, 2 \},$$

the forms $a(\cdot, \cdot)$, $a_1(\cdot, \cdot)$ are defined in section 6,

$$l(T; z_1) = \int_\Omega r^2 (3\lambda + 2\mu) \alpha T \left(\frac{\partial z_1}{\partial r} + \frac{2 z_1}{r} \right) dr - 2\xi^2 \overline{p} z_1(\xi) + r_1^2 p_1 z_1(r_1) - r_2^2 p_2 z_1(r_2),$$

$$l_1(u; z_2) = \int_{r_1}^{r_2} r^2 \overline{f} z_2 \, dr + r_1^2 u z_2(r_1) + \beta r_2^2 z_2(r_2) + \xi^2 \frac{R_2 \omega}{R_1 + R_2} [z_2] - \xi^2 \omega z_2^+.$$

Functional discrepancy has the form (9). We shall solve the problem (114), (9) by means of gradient methods (13). The expressions of (17)–(19) type take place.

For every approximation u_n of solution $u \in \mathcal{U}$ of the problem (114), (9) we write the conjugate problem as:

$$-(\lambda + 2\mu) \left(\frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) - 2\psi \right) = 0, \ (r, t) \in \Omega_T,$$

$$-r^2 c \frac{\partial p}{\partial t} - \frac{\partial}{\partial r} \left(r^2 k \frac{\partial p}{\partial r} \right) - r^2 (3\lambda + 2\mu) \alpha \left(\frac{\partial \psi}{\partial r} + \frac{2\psi}{r} \right) = 0, \ (r, t) \in \Omega_T,$$

$$\sigma_r(\psi)\big|_{r=r_1} = 0, \ \sigma_r(\psi)\big|_{r=r_2} = \frac{1}{r_2^2} (y(u_n; r_2, t) - f_0),$$

$$-k \frac{\partial p}{\partial r}\bigg|_{r=r_1} = 0, \ k \frac{\partial p}{\partial r}\bigg|_{r=r_2} = -\overline{\alpha} p(r_2, t), \tag{115}$$

$$[\psi]\big|_{r=\xi} = 0, \ [\sigma_r(\psi)]\big|_{r=\xi} = 0,$$

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$$\left[k\frac{\partial p}{\partial r}\right]\bigg|_{r=\xi} = 0, \quad \left\{k\frac{\partial p}{\partial r}\right\}^{\pm} = \frac{[p]}{R_1 + R_2},$$

$$p\big|_{t=\overline{T}} = 0, \quad r \in \overline{\Omega}.$$

Definition 16. We call as the generalized solution of the initial boundary value problem (115) the vector-function $Y^* = (\psi, \, p) \in V_d$, which $\forall z = (z_1, z_2) \in V_0$ holds the system of relations:

$$a(\psi, z_1) = (y(u_n; r_2, t) - f_0) z_1(r_2), \quad t \in (0, \overline{T}), \tag{116}$$

$$-\left(r^2 c \frac{\partial p}{\partial t}, z_2\right) + a_1(p, z_2) - \int_{\Omega} r^2 (3\lambda + 2\mu) z_2 \alpha\left(\frac{\partial \psi}{\partial r} + \frac{2\psi}{r}\right) dr = 0, \quad t \in (0, \overline{T}), \tag{117}$$

$$(r^2 c p, z_2)(\overline{T}) = 0, \tag{118}$$

where

$$V_d = \left\{ v \in \overline{V} : [v_1]\big|_{r=\xi} = 0, t \in (0, \overline{T}), \sum_{i,j=1}^{2} \int_0^{\overline{T}} \|v_i\|_{W_2^1(\Omega_j)}^2 dt < \infty, \right.$$

$$\left. \frac{\partial v_2}{\partial t}\bigg|_{\Omega_i} \in L^2(0, \overline{T}; L_2(\Omega_i)), i = 1, 2 \right\}.$$

If we select in the relation (116) instead of the function z_1 the difference $y(u_{n+1}) - y(u_n)$ and the difference $T(u_{n+1}) - T(u_n)$ instead of z_2 in (117), (118), taking into account (114) we have

$$\langle J'_{u_n}, \Delta u_n \rangle = (\overline{y}(u_n) - f_0, \overline{y}(u_{n+1}) - \overline{y}(u_n))_{L_2} =$$

$$= \int_0^{\overline{T}} a(\psi, y(u_{n+1}) - y(u_n)) \, dt - \int_0^{\overline{T}}\int_{\Omega} r^2 (3\lambda + 2\mu)(T(u_{n+1}) - T(u_n)) \alpha\left(\frac{\partial \psi}{\partial r} + \frac{2\psi}{r}\right) dr \, dt +$$

$$+ \int_0^{\overline{T}} a_1(p, T(u_{n+1}) - T(u_n)) \, dt = \int_0^{\overline{T}} \Delta u_n r_1^2 p(r_1, t) \, dt. \tag{119}$$

On the basis of (119) we obtain $J'_{u_n} = \tilde{\psi}_n$, where $\tilde{\psi}_n = r_1^2 p(r_1, t)$, $\|J'_{u_n}\|^2 = \int_0^{\overline{T}} \tilde{\psi}_n^2 dt$.

The presence of gradient J'_{u_n} makes it possible to use gradient methods (13) for determination of the $(n+1)$-th approximation u_{n+1} of the solution $u \in \mathcal{U}$ of the problem (114), (9).

10. Simultaneous identification of density of heat flow and source

Let on the domain Ω_T $(\Omega = \Omega_1 \cup \Omega_2)$ the equation of elastic equilibrium (109) is defined, and variation of temperature T holds the equation (110). At the ends of segment $[r_1, r_2]$ $\forall t \in (0, \overline{T})$ boundary conditions are given

$$\sigma_r(y)\big|_{r=r_i} = -p_i, \quad i = 1, 2,$$

$$-k\frac{\partial T}{\partial r}\bigg|_{r=r_1} = u_1, \quad k\frac{\partial T}{\partial r}\bigg|_{r=r_2} = -\overline{\alpha}T(r_2, t) + \beta. \tag{120}$$

At the point $r = \xi$ $\forall t \in (0, \overline{T})$ conjugation conditions have the form:

$$[y] = \delta, \ \{\sigma_r(y)\}^- = -\overline{p}, \ \{\sigma_r(y)\}^+ = \overline{p},$$

$$R_1\left\{k\frac{\partial T}{\partial r}\right\}^- + R_2\left\{k\frac{\partial T}{\partial r}\right\}^+ = [T], \tag{121}$$

$$\left[k\frac{\partial T}{\partial r}\right] = u_2.$$

Let us specify for $t = 0$ the initial condition (98). We assume that at points d_i, $i = \overline{0, N}$, displacements are known and are given by the equalities

$$y(d_i, t) = f_i(t), \ i = \overline{0, N}, \ t \in (0, \overline{T}), \tag{122}$$

where $d_0 = r_2$, $d_i \in \Omega, i = \overline{1, N}$.

We obtained the problem (109), (110), (120)–(122), (98), which consists in determination of a vector $u = (u_1, u_2) \in \mathcal{U} = C([0, \overline{T}]) \times C([0, \overline{T}])$, for which the first component y of the classical solution $Y = (y, t)$ of the initial boundary value problem (109), (110), (120),(121), (98) holds the equalities (122). Instead of the classical solution $Y = (y, T)$ of the initial boundary value problem (109), (110), (120), (121), (98) we shall use its generalized solution.

Definition 17. For every fixed $u \in \mathcal{U}$ we call as the generalized solution of the initial boundary value problem (109), (110), (120), (121), (98) the vector-function $Y = (y, T) \in V$, which $\forall z = (z_1, z_2) \in V_0$ holds the system of relations

$$a(y, z_1) = l(T; z_1), \ \forall t \in (0, \overline{T}), \tag{123}$$

$$\left(r^2 c\frac{\partial T}{\partial t}, z_2\right) + a_1(T, z_2) = l_1(u; z_2), \forall t \in (0, \overline{T}), \tag{124}$$

$$(r^2 cT, z_2)(0) = (r^2 c\overline{T}_0, z_2), \tag{125}$$

where the sets V, V_0 are defined in section 9, and the bilinear forms $a(\cdot, \cdot)$, $a_1(\cdot, \cdot)$ are specified in section 6,

$$l(T; z_1) = \int_\Omega r^2(3\lambda + 2\mu)\alpha T\left(\frac{\partial z_1}{\partial r} + \frac{2z_1}{r}\right)dr - 2\xi^2 \overline{p}z_1(\xi) + r_1^2 p_1 z_1(r_1) - r_2^2 p_2 z_1(r_2),$$

$$l_1(u; z_2) = \int_{r_1}^{r_2} r^2\overline{f}z_2 dr + r_1^2 u_1 z_2(r_1) + \beta r_2^2 z_2(r_2) + \frac{\xi^2 R_2 u_2}{R_1 + R_2}[z_2] - \xi^2 u_2 z_2^+.$$

The functional-discrepancy has the form (51). We shall solve the problem (51), (123)–(125) by means of gradient methods (13).

For every approximation u_n of solutions $u \in \mathcal{U}$ of the problem (51), (123)–(125) we write the conjugate problem as

$$-(\lambda + 2\mu)\left(\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) - 2\psi\right) = 0, \ (r, t) \in \Omega_{d_T},$$

$$-r^2 c\frac{\partial p}{\partial t} - \frac{\partial}{\partial r}\left(r^2 k\frac{\partial p}{\partial r}\right) - r^2(3\lambda + 2\mu)\alpha\left(\frac{\partial\psi}{\partial r} + \frac{2\psi}{r}\right) = 0, \ (r, t) \in \Omega_{d_T},$$

$$\sigma_r(\psi)\big|_{r=r_1} = 0, \; \sigma_r(\psi)\big|_{r=r_2} = \frac{1}{r_2^2}(y(u_n; r_2, t) - f_0),$$

$$-k\frac{\partial p}{\partial r}\bigg|_{r=r_1} = 0, \; k\frac{\partial p}{\partial r}\bigg|_{r=r_2} = -\overline{\alpha}p(r_2, t),$$

$$[\psi]\big|_{r=d_i} = 0, \; [\sigma_r(\psi)]\big|_{r=d_i} = -\frac{1}{d_i^2}(y(u_n; d_i, t) - f_i), \; i = \overline{1, N}, \tag{126}$$

$$[\psi]\big|_{r=\xi} = 0, \; \{\sigma_r(\psi)\}^- = \{\sigma_r(\psi)\}^+ = 0,$$

$$[p]\big|_{r=d_i} = 0, \; \left[k\frac{\partial p}{\partial r}\right]\bigg|_{r=d_i} = 0, \; i = \overline{2, N},$$

$$\left[k\frac{\partial p}{\partial r}\right]\bigg|_{r=\xi} = 0, \; \left\{k\frac{\partial p}{\partial r}\right\}\bigg|_{r=\xi}^{\pm} = \frac{[p]\big|_{r=\xi}}{R_1 + R_2}, \; t \in (0, \overline{T}),$$

$$p\big|_{t=\overline{T}} = 0, \; r \in \overline{\Omega},$$

where $\Omega_d = \Omega \setminus \gamma_d$, $\gamma_d = \bigcup\limits_{i=1}^{N} d_i$, $\Omega_{d_T} = \Omega_d \times (0, \overline{T})$, $\sigma_r(\psi) = (\lambda + 2\mu)\frac{\partial \psi}{\partial r} + 2\lambda\frac{\psi}{r}$.

Definition 18. We call as the generalized solution of the initial boundary value problem (126) the vector-function $Y^* = (\psi, p) \in V_d$, which holds $\forall z = (z_1, z_2) \in V_d^0$ the system of relations

$$a(\psi, z_1) = \sum_{i=0}^{N}(y(u_n; d_i, t) - f_i)z_1(d_i), \; t \in (0, \overline{T}), \tag{127}$$

$$-\left(r^2 c\frac{\partial p}{\partial t}, z_2\right) + a_1(p, z_2) - \int\limits_{\Omega} r^2(3\lambda + 2\mu)z_2\alpha\left(\frac{\partial \psi}{\partial r} + \frac{2\psi}{r}\right)dr = 0, \; t \in (0, \overline{T}), \tag{128}$$

$$(r^2 cp, z_2)(\overline{T}) = 0. \tag{129}$$

If we select in the relation (127) instead of the function z_1 the difference $y(u_{n+1}) - y(u_n)$ and the difference $T(u_{n+1}) - T(u_n)$ instead of z_2 in (128), (129), taking into account (123)–(125) we have

$$\left\langle J'_{u_n}, \Delta u_n \right\rangle = (\overline{y}(u_n) - f_0, \overline{y}(u_{n+1}) - \overline{y}(u_n))_{L_2} =$$

$$= \int\limits_0^{\overline{T}} a(\psi, y(u_{n+1}) - y(u_n))\,dt - \int\limits_0^{\overline{T}}\int\limits_{\Omega} r^2(3\lambda + 2\mu)(T(u_{n+1}) - T(u_n))\alpha\left(\frac{\partial \psi}{\partial r} + \frac{2\psi}{r}\right)dr\,dt +$$

$$+ \int\limits_0^{\overline{T}} a_1(T(u_{n+1}) - T(u_n), p)\,dt = \int\limits_0^{\overline{T}}(l_1(u_{n+1}; p) - l_1(u_n; p))\,dt =$$

$$= \int\limits_0^{\overline{T}} \Delta u_{1n}r_1^2 p(r_1, t)\,dt + \int\limits_0^{\overline{T}} \Delta u_{2n}\left(\frac{R_2}{R_1 + R_2}[p] - p^+\right)\xi^2\,dt.$$

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Hence, $J'_{u_n} = \tilde{\psi}_n$, where

$$\tilde{\psi}_n = \{\tilde{\psi}_n^i\}_{i=1}^2, \quad \tilde{\psi}_n^1 = r_1^2 p(r_1, t),$$

$$\tilde{\psi}_n^2 = -\frac{R_1 p^+ + R_2 p^-}{R_1 + R_2} \xi^2, \quad \|J'_{u_n}\|^2 = \sum_{i=1}^2 \int_0^{\overline{T}} (\tilde{\psi}_n^i)^2 dt. \tag{130}$$

The presence of gradient J'_{u_n} makes it possible to use gradient methods (13) for determination of the $(n+1)$-th approximation u_{n+1} of solution $u \in \mathscr{U}$ of the problem (123)–(125), (51).

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Identifying Failures in Sensors of a Spacecraft Attitude Control System

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ABSTRACT

A problem of failure detection and identification in sensors of a spacecraft attitude control system under condition of their minimum number or minimum redundancy is investigated. The effectiveness of procedures under consideration is illustrated via angular rate sensors and Sun sensor as examples.

Key words: spacecraft attitude control system, minimum redundancy, angular rate sensor, Sun sensor.

Introduction

For a spacecraft (SC) that moves along a circular or weakly elliptical orbit consideration is given to the problem of single failure detection and identification in a set of sensing elements (sensors) of spacecraft (SC) attitude control system. Unlike the known solutions requiring for identification of a failed sensor not less than five one-type devices, the paper in question suggests the procedure of detecting a single failure by using smaller number of actively operating sensors. This effect is attained by involving a data on SC motion parameters from the previous step of identification and arranging "virtual" measurements. Operability of proposed procedures of failures identification is illustrated by results of modeling the process of identifying failures of angular rate measurers and Sun position sensor.

To basic directions of using an artificial intelligence in vehicle-borne segment of control system of some moving objects one generally refers control and diagnosis of state of control system facilities and other subsystems of controlled object [1]. The problems of this direction are related to increase of reliability of control system operation attained via forecasting a state of subsystems and implementing an operation principle by the real state of system.

The information subsystem of SC attitude control system contains a set of sensors for obtaining information required for calculating and correcting SC attitude parameters. The measurement set can include angular rate sensors (ARS), triaxial magnetometer, Sun position sensor and other facilities. The sensor failure generally implies such its state that the measurement error exceeds a certain admissible level N. Solving the problem of the above subsystem insensibility to sensors failures is feasible through duplicating the subsystem as a whole or reserving sensors and control of redundant data.

The increase of primary information reliability due to reservation of sensors and control of redundant data is related to sufficiently deep reservation of each type sensors. So, for identifying k simultaneous failures the number n of sensors in a unit while one-time measurement of three dimensional vector (for example, the angular rate vector) should satisfy the condition [2] $n \geq 2k+3.$. From this relation it follows that while identifying a single failure the measurement unit is to contain not less than five sensors. However, in practice there could be a situation when it is necessary to detect and identify in a real time a single failure in nonredundant sensor unit or in conditions of their minimum redundancy.

For the above situation the given paper presents the technique and algorithm of identifying failures of angular rate sensors. Consideration is also given to procedure of detecting Sun sensor failure in a set of sensing elements.

Algorithms of failure identification

For synthesis of failures identification algorithms in a unit of sensing elements we make use of parity space method [3–6]. Its essence consists in controlling consistency of equations of the system (relations of analytical redundancy) by using results of real measurements. There exist two forms of relations of analytical redundancy [5]: algebraic relations between measurements of redundant sensors and relations in the form of difference or differential equations. We consider both variants in detail.

Let in the right orthogonal system of coordinates xyz rigidly related with the object there is installed the redundant unit of sensing elements of n sensors for measuring three-dimensional vector $\mathbf{x}$. It is assumed that sensitivity axes of any three measurers do not lie in the same plane. With sensors dynamics neglected, the output $\mathbf{y}$ of the unit is related to the measured quantity $\mathbf{x}$ by the relation

$$\mathbf{y} = A\mathbf{x} + \mathbf{e}, \tag{1}$$

in which $A = \{a_{ij}\}$ is $(n \times 3)$-matrix of direction cosines of angles between sensitivity axes of sensors and coordinate axis x, y, z; $\mathbf{e} = \{e_i\}$ $(i = 1, \ldots, n)$ is the vector of measurements error.

The senor numbered i is considered operable if $|e_i| < N$. The failure identification suggests the process of identifying the failed sensor and the estimated error of its measurement.

Now we show, that with the available information on the output $\mathbf{y} = \{y_i\}$ $(i = 1,\ldots, n; n \geq 5)$ of the measurer unit and the matrix A of direction cosines of axes of sensor sensitivity we synthesize the identification algorithm of sensors failures.

The existence of matrices V satisfying the conditions

$$VA = 0,$$

$$V^{\mathrm{T}}V = I_n - A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}, \tag{2}$$

$$VV^{\mathrm{T}} = I_m$$

is known [3–5]. These matrices allow one to present the vector $\mathbf{p} = V\mathbf{y}$, called the parity vector in the form

$$\mathbf{p} = V\mathbf{e}. \tag{3}$$

The number m of possible linearly independent equations (3) equals the difference between the number of sensors n and the dimensionality of the quantity being measured [4] (in the case of three-dimensional vector $\mathbf{x}$ $m = n - 3$).

Since the rank of matrix V in (3) equals m, then for the known vector $\mathbf{p}$ the equations (3) have the infinite set of solutions which can be written in the form

$$\mathbf{e} = \mathbf{e}_* + \mathbf{E}. \tag{4}$$

Here $\mathbf{e}_*$ is one of the solutions satisfying the equation (3) and the vector $\mathbf{E}$ belongs to the kernel X of operator V:

$$\mathbf{E} \in X = \{\mathbf{e} : V\mathbf{e} = 0\}.$$

We determine the vector $\mathbf{E}$ in (4) in the form of relation

$$\mathbf{E} = (I_n - V^{\mathrm{T}}V)\mathbf{l}, \tag{5}$$

in which $\mathbf{l}$ is the arbitrary n-dimensional vector.

By substituting (5) in (4) from the set of vectors $\mathbf{e}$ by the corresponding selection of $\mathbf{l}$ we find such vector $\mathbf{e}_0$ the norm of which is minimal. It is determined due to minimizing the functional

$$J_0 = \|\mathbf{e}\|^2 = \|\mathbf{e}_* + (I_n - V^{\mathrm{T}}V)\mathbf{l}\|^2.$$

Taking into consideration that $I_n - V^{\mathrm{T}}V$ in equality (5) is the projection matrix, the vector $\mathbf{l}$ bringing the minimum value to the functional J_0 is calculated by the formula

$$\mathbf{l} = -(I_n - V^{\mathrm{T}}V)\mathbf{e}_*. \tag{6}$$

The substitution of (6) in (4), (5) yields the required solution:

$$\mathbf{e}_0 = G\mathbf{e}_*, \quad G = V^{\mathrm{T}}V. \tag{7}$$

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It is important to note, that for the given vector $\mathbf{p}$ the relation (7) is the mapping of set X_* of vectors $\mathbf{e}_*$ into the vectors $\mathbf{e}_0$ of minimum length.

From (7) it follows that calculation of vector $\mathbf{e}_0$ assumes one of the solutions of equation (3) to be known. In [6] to find the solution $\mathbf{e}_*$ one employed the method of linear programming. However, the vector $\mathbf{e}_0$ of minimum length can be obtained via pseudoinversion of matrix V in the equation (3) [7]. Indeed,

$$\mathbf{e}_0 = V^+ \mathbf{p}, \ V^+ = V^{\mathrm{T}}(VV^{\mathrm{T}})^{-1}$$

or considering the third equality in (2) we have

$$\mathbf{e}_0 = V^{\mathrm{T}}\mathbf{p}. \tag{8}$$

If we consider (8) as one of the possible solutions $\mathbf{e}_*$ of equation (3), then substitution of (8) into (7) naturally leads to the identity.

If the vector $\mathbf{p}$ contains only information about the error $e_i = \rho_i$ of the i-th sensor (small random errors being neglected), then the set X_* also contains the vector $\mathbf{e}_{*,i} = [0\,0\ldots\rho_i\,\ldots 0]^{\mathrm{T}}$.

Of elements g_{ik} $(i, k = 1,..., n)$ of the projection matrix G we form the vectors

$$\mathbf{g}_k = [g_{1k}\ g_{2k}\ \ldots g_{nk}]^{\mathrm{T}}\ \ (k = 1,\ldots, n).$$

Assuming the vector $\mathbf{e}_0$ to be known we find the minimum value of functional

$$J(k) = \left\|\mathbf{q}_k\right\|^2, \ \ \mathbf{q}_k = \rho_k \mathbf{g}_k - \mathbf{e}_0 \tag{9}$$

and its realizing value of ρ_k of sensor error for each of n vectors $\rho_k \mathbf{g}_k$. As a result we have the set n of quantities

$$\rho_k = \frac{\mathbf{g}_k^{\mathrm{T}}\mathbf{e}_0}{\left\|\mathbf{g}_k\right\|^2}\ \ (k = 1,\ldots, n) \tag{10}$$

and their corresponding values of $J(k)$ $(k = 1,\ldots, n)$ of the functional (9). The value of $k = k_*$ for which the condition $k_* = \arg\min J(k)$ holds true corresponds to number of required sensor; its error ρ_k is estimated by the expression (10).

We sum up the sequence of actions for realization of failure identification algorithm.

Preliminary (by the known matrix A) one calculates the matrix V with properties of (2) and $n \times n$ projection matrix G.

The process of failure identification is performed as follows:

— by the output $\mathbf{y}$ of the measurer unit and (8) one forms the vector $\mathbf{e}_0$ of minimum length;

— by formulae (10) one calculates the set of n quantities ρ_k and their corresponding values of $J(k)$ of functional (9);

— the number k, for which the value of functional $J(k)$ (or the length of vector $\mathbf{q}_k$) is minimum corresponds to the number of sensor with the estimated error ρ_k. When $|\rho_k| > N$ the failure of the k-th sensor is detected.

The above method of parity space applied to algebraic relations (1) describing the measurement process has been generalized in [8, 9] for the case of using redundant relations given by differential or difference equations.

Let the dynamic system be described by the linear stationary discrete equations in the space of states

$$\mathbf{x}(k+1) = A\mathbf{x}(k) + B\mathbf{u}(k),$$
$$\mathbf{y}(k) = C\mathbf{x}(k),$$

(11)

where k is the discrete time; $\mathbf{x}$, $\mathbf{u}$ and $\mathbf{y}$ are the vectors of state, control and output of the system with the dimensions n, p and q correspondingly; A, B, C are the matrices of corresponding sizes.

We now determine the subspace of $(\mu+1)q$-dimensional vectors $\mathbf{v}$ by the relation

$$P = \left\{ \mathbf{v} : \mathbf{v}^{\mathrm{T}} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\mu} \end{bmatrix} = 0 \right\}.$$

(12)

The space P is called the parity space of order μ [8].

According to [5] each vector $\mathbf{v}$ from (12) at any time instant k can be used for parity control performed by the formulae

$$r(k) = \mathbf{v}^{\mathrm{T}} \left[\begin{bmatrix} \mathbf{y}(k-\mu) \\ \vdots \\ \mathbf{y}(k) \end{bmatrix} - H \begin{bmatrix} \mathbf{u}(k-\mu) \\ \vdots \\ \mathbf{u}(k) \end{bmatrix} \right],$$

(13)

$$H = \begin{bmatrix} 0 \\ CB & 0 & & \mathbf{0} \\ CAB & CB & 0 \\ \vdots \\ CA^{\mu-1}B & \dots & \dots & CAB & CB & 0 \end{bmatrix}.$$

The described approach allows one to single out the most reliable relations and thereby to create the robust procedure of failures detecting and localizing.

The question on selecting the order μ of parity space P has been discussed in [5, 9].

Identification of failures of angular rate sensors

Let the information subsystem of SC attitude control system contains the set of angular rate measurers of four identical devices — the angular rate sensors. Unit vectors of measuring axes of equipment in a related with SC system of coordinates $Oxyz$, whose origin O coincides with the object mass center is written in the form $\mathbf{n}_i = \{\alpha_i, \beta_i, \gamma_i\}$ $(i = x, y, z, r)$

Under assumption that three of the above sensors operate in a design mode and the fourth is the redundant one, then the real number of actively operating ARS equals three. According to the above results such number of sensors is insufficient not only for localizing ARS failure but detecting the failure occurrence as well. With four actively operating sensors one can only identify the fact of failure. In such

situation to counteract the disturbances acting on SC due to possible single errors or ARS failures it is necessary to apply additional data aimed at forming not less than five independent measurements of SC angular rate vector. It could be the information from other sensors of information subsystem (for example, magnetometer or Sun position sensor indirectly containing data on SC absolute angular rate) or some other data on SC angular motion.

Combination of readings of three or four ARS with those of one of the above devices allows one (under certain conditions) to solve the problem of detecting and identifying ARS failure. However, on information processing step it imposes certain burden on SC computational complex.

We now make use of procedure of identifying failures based on parity space method; as additional information we apply the value of SC angular rate vector from the previous measurement step. For diagnosing and localizing the failure of one of actually operating ARS we form a measuring structure of three ARS and two additional "virtual" devices. The measuring axes of "virtual" devices are to be selected so that thereby obtained matrix A in relations (1) would satisfy the requirements on solvability of the problem considered.

Let while diagnosing the failures of standard ARS of measuring subsystem (x-, y- and z-ARS) the sensitivity axis of the first "virtual" sensor coincides with the measuring axis of redundant gyroscope (r-ARS). Then in matrix A from the equality (1) which in the considered case has the structure of the form

$$A = \begin{bmatrix} \alpha_x & \beta_x & \gamma_x \\ \alpha_y & \beta_y & \gamma_y \\ \alpha_z & \beta_z & \gamma_z \\ \alpha_r & \beta_r & \gamma_r \\ \alpha & \beta & \gamma \end{bmatrix}, \tag{14}$$

it is necessary to select the coordinates α, β, γ of the unit vector $\mathbf{n}$ of the sensitivity axis of the second "virtual" ARS so that none of three measuring axes of thereby formed "unit" of sensing elements would not lie in the same plane.

If α, β, γ are determined, then as the output $\mathbf{y}$ (the result of measuring the "unit" of ARS) we take the vector

$$\mathbf{y} = [y_1, y_2, y_3, y_4^*, y_5^*]^T,$$
$$y_4^* = \mathbf{n}_r^T \boldsymbol{\omega}_{n-1}, \quad y_5^* = \mathbf{n}^T \boldsymbol{\omega}_{n-1} \tag{15}$$

($\boldsymbol{\omega}_{n-1}$ — is the SC angular rate vector from the previous measurement step). This data processing by the algorithm of detection and identification of failures whose realization has been described above solves the stated problem.

With failure of one of standard ARS and substitution of its readings by those of redundant gyroscope the failure identification scheme in a new configuration of measurer unit undergoes minimum changes: as the measuring axis of the first virtual sensor we take the axis of sensitivity of the failed gyroscope and the output (15) changes. For example, if we eliminate the failed y-ARS with the unit vector of measuring axis $\mathbf{n}_y = \{\alpha_y, \beta_y, \gamma_y\}$ the vector $\mathbf{y}$ is determined by the relation

$$\mathbf{y} = [y_1, y_2^*, y_3, y_4, y_5^*]^T,$$
$$y_2^* = \mathbf{n}_y^T \boldsymbol{\omega}_{n-1}.$$

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In the case of four actually operating ARS as the output of measurer "unit" we take the vector $\mathbf{y} = [y_1, y_2, y_3, y_4, y_5^*]^\mathrm{T}$.

The requirements to be satisfied by the elements α, β, γ of matrix A in expression (14) are formulated by the results presented in [10]. According to [10], the problem of failures detection and identification is solvable if any matrices A_{pq} $(p, q = 1,\ldots, 5; p \neq q)$ obtained from A by simultaneous crossing out the rows with numbers p and q are not degenerated.

It is worth noting, that for stable identification of failures the determinants of matrices V_{pq} are to be taken the largest by module. It is attained by variation of the parameters α, β, γ from the set Q whose structure is defined by the structure of matrices A_{pq}.

The substitution of angular rate vector $\boldsymbol{\omega}$ by its estimate $\boldsymbol{\omega}_{n-1}$ for further application in formation of readings of the k-th "virtual" ARS with the unit vector of measuring axis $\mathbf{n}_k$ is followed by occurrence of systematic error e_k on output of this "ARS" with the estimate

$$e_k = \left| \mathbf{n}_k^\mathrm{T} (\boldsymbol{\omega}_{n-1} - \boldsymbol{\omega}) \right|. \tag{16}$$

Hence, the threshold N should be consistent with a priori known level of measurements noises and the estimate (16) in order not to miss the sensor failure or not to admit the malfunction of failure identification algorithm.

Simulation of identification algorithm of angular rate sensors failures

We specify the value of matrix A determined by formula (14). We assume, that the measuring axes of standard ARS are parallel to axes of related system of coordinates $Oxyz$ and are equally directed, the unit vector $\mathbf{n}_r$ of the measuring axis of a redundant device is determined by the relation $\mathbf{n}_r = [-1/\sqrt{3} \ -1/\sqrt{3} \ -1/\sqrt{3}]^\mathrm{T}$. In this case the requirement for solvability of problem of failures identification is reduced to the coordinates α, β, γ of unit vector $\mathbf{n}$ of virtual gyroscope measuring axis having to belong to the set

$$Q = \{\alpha, \beta, \gamma : \alpha \neq 0, \beta \neq 0, \gamma \neq 0, \alpha \neq \beta, \alpha \neq \gamma, \beta \neq \gamma, \alpha^2 + \beta^2 + \gamma^2 = 1\}. \tag{17}$$

Operability of proposed identification algorithm of failures in the angular rate sensor unit was investigated using fragments of recording angular rates of micro satellite type SC rotation in the mode of orbital attitude. Stabilization of SC motion was implemented with accuracy of order 5° in attitude and not worse than 0.01 °/s — in angular rate. The stabilization process was imitated by mathematical simulation of dynamics of controlled motion of SC moving along the orbit close to circular one $(\omega_0 \approx 0.0010731 \, \mathrm{s}^{-1})$.

Periodically in reading of one of actually operating ARS there was introduced a replacement constant in quantity (sensor failure).

Failures identification was carried out with a cycle of 1 s for the following values of parameters involved in the algorithm of failures detection and localization:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} \\ -0.21132 & -0.57735 & 0.78868 \end{bmatrix},$$

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$$V = \begin{bmatrix} 0.40825 & 0{,}40825 & 0.40825 & 0.70711 & 0 \\ 0.14943 & 0.40825 & -0.55768 & 0 & 0.70711 \end{bmatrix},$$

$$G = \begin{bmatrix} 0.18900 & 0.22767 & 0.08333 & 0.28868 & 0.10566 \\ 0.22767 & 0.33333 & -0.06100 & 0.28868 & 0.28868 \\ 0.08333 & -0.06100 & 0.47767 & 0.28868 & -0.39434 \\ 0.28868 & 0.28868 & 0.28868 & 0.5 & 0 \\ 0.10566 & 0.28868 & -0.39434 & 0 & 0.5 \end{bmatrix}.$$

The threshold N was assumed to equal $0.0015\,°/s$. There were performed some series of failures testing: one of them foresaw the failures identification of standard gyroscopes of information subsystem; in another series one of the standard gyroscopes was substituted by the redundant (r-ARS); consideration was given to different time intervals of SC motion. In different variants of failure simulation a reading of one of gyroscopes was supplemented with equal in quantity but different in sign displacements in $0.002\,°/s$.

Typical results of simulation are presented in Table 1. In columns referring to different configurations of unit of actual ARS for five measuring channels there are the estimates ρ_k ($°/s$) of reading displacement of the corresponding sensor and the norm $\|\mathbf{q}_k\|$ ($1/s$) of vectors $\mathbf{q}_k$ (see formulae (9), (10)). Minimum values of $\|\mathbf{q}_k\|$ and their corresponding displacements of ARS readings are given in medium type. The result of operation of failures identification algorithm is presented in the last row of column.

Table 1

Parameters	Sets of sensors			
	x-, y-, z-ARS	y-, z-, r-ARS	x-, z-, r-ARS	x-, y-, r-ARS
$\|\mathbf{q}_k\|$	$0.842{\cdot}10^{-5}$	$0.249{\cdot}10^{-4}$	$\mathbf{0.146{\cdot}10^{-5}}$	$0.847{\cdot}10^{-5}$
	$\mathbf{0.430{\cdot}10^{-5}}$	$0.255{\cdot}10^{-4}$	$0.610{\cdot}10^{-5}$	$0.154{\cdot}10^{-4}$
	$0.189{\cdot}10^{-4}$	$\mathbf{0.133{\cdot}10^{-5}}$	$0.142{\cdot}10^{-4}$	$0.210{\cdot}10^{-4}$
	$0.138{\cdot}10^{-4}$	$0.214{\cdot}10^{-4}$	$0.384{\cdot}10^{-5}$	$\mathbf{0.406{\cdot}10^{-6}}$
	$0.133{\cdot}10^{-4}$	$0.140{\cdot}10^{-4}$	$0.147{\cdot}10^{-4}$	$0.257{\cdot}10^{-4}$
ρ_k	0.0023	-0.0008	$\mathbf{-0.0020}$	0.0032
	$\mathbf{0.0019}$	0.0001	-0.0014	0.0021
	-0.0003	$\mathbf{-0.0021}$	-0.0005	0.0012
	0.0011	-0.0011	-0.0012	$\mathbf{0.0021}$
	0.0011	0.0017	-0.0003	0.0001
Displacement	0.0019	-0.0021	-0.0020	0.0021
	y-ARS failure	z-ARS failure	x-ARS failure	r-ARS failure

The analogous results hold while testing failures in a unit of four simultaneously operating angular rate sensors.

Analysis of simulation results suggests the ability of the proposed procedure to identify and localize single failures in a unit of angular rate sensors by decreasing the number of actual devices to three.

Identification of failures of Sun position sensor

Let the Sun position sensor (Sun sensor) be related with the right orthogonal system of coordinates $x_S y_S z_S$ with the origin in the center of projecting the optical system and the axis y_S coinciding with the optical axis of Sun sensor. The axes of system of coordinates $x_S y_S z_S$ are assumed to coincide in direction with the corresponding axes of the system of coordinates $Oxyz$ related with SC.

Equations of Sun motion on the sensor image plane can be obtained by circular permutation of corresponding coordinates and indices in evolution equations of point object described in [11] for the case when the optical axis of the system is directed along the axis z_S. These equations are of the form

$$\dot{x} = \frac{1}{F} xz\omega_x - z\omega_y - F\left(1 + \frac{x^2}{F^2}\right)\omega_z,$$

$$\dot{z} = F\left(1 + \frac{z^2}{F^2}\right)\omega_x + x\omega_y - \frac{1}{F} xz\omega_z, \tag{18}$$

where x and z are the coordinates of Sun center on the plane $x_S z_S$ of coordinate system $x_S y_S z_S$; F is the focal distance of optical system; ω_x, ω_y, ω_z are the projections of vector of SC absolute angular rate on the axis of related system of coordinates.

The SC motion is assumed to occur on a circular orbit in a mode close to mode of three-axis attitude in the orbital coordinate system. In this case the system of equations (18) is simplified and can be presented in the form of relations

$$\dot{x} = -\omega_0 z - F\omega_z, \quad \dot{z} = \omega_0 x + F\omega_x \tag{19}$$

(ω_0 is the angular rate of SC orbital motion).

Assuming the angular rates ω_x and ω_z to be constant on the step h of time quantization of the system of equations (19) we put down these equations in the form (11)

$$\mathbf{x}(k+1) = A\mathbf{x}(k) + B\mathbf{u}(k),$$

$$\mathbf{y}(k) = C\mathbf{x}(k),$$

where the following notations are introduced:

$$A = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}, \ B = \{b_{ij}\} = \frac{F}{\omega_0}(A - E), \ C = E, \ \mathbf{x}(k) = \begin{bmatrix} x(k) \\ z(k) \end{bmatrix}, \ \mathbf{u}(k) = \begin{bmatrix} \omega_x(k) \\ \omega_z(k) \end{bmatrix}, \tag{20}$$

$$c = \cos\omega_0 h, \ s = \sin\omega_0 h \ (i, j = 1, 2)$$

(E is the unit matrix).

We now introduce the space of parity P of order 1 ($\mu = 1$) — the space of four dimensional vectors $\mathbf{v} = [v_1, ..., v_4]^T$. Taking into consideration the relation (12) and structure of matrices A and C from (20) we define this space by the expression

$$P = \left\{ \mathbf{v} : \mathbf{v}^{\mathrm{T}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ c & -s \\ s & c \end{bmatrix} = 0 \right\}.$$

After simple transformations we put it down in the form

$$P = \{ \mathbf{v} : v_3 = -cv_1 + sv_2, \ v_4 = -sv_1 - cv_2 \}. \tag{21}$$

As it was already mentioned, each vector $\mathbf{v}$ of the set (21) at any time instant k can be applied to parity control performed by formula (13). Since in the considered case the matrix

$$H = \begin{bmatrix} O & O \\ B & O \end{bmatrix}$$

(O is zero (2×2)-matrix), then $r(k)$ from (13) is determined by equality

$$r(k) = \mathbf{v}^{\mathrm{T}} \left\{ \begin{bmatrix} \mathbf{y}(k-1) \\ \mathbf{y}(k) \end{bmatrix} - \begin{bmatrix} O \\ B\mathbf{u}(k) \end{bmatrix} \right\} =$$

$$= v_1 x(k-1) + v_2 z(k-1) + v_3 [x(k) - b_{11}\omega_x(k-1) - b_{12}\omega_z(k-1)] +$$

$$+ v_4 [z(k) - b_{21}\omega_x(k-1) - b_{22}\omega_z(k-1)]. \tag{22}$$

It is easy to show that in the absence of noises and operable Sun sensor $r(k)$ in (22) vanishes. With measurements on noise background or with failure occurred on optical system (i.e., $x(k) = x_*(k) + \Delta x(k)$, $z(k) = z_*(k) + \Delta z(k)$, where $x_*(k), z_*(k)$ are accurate values of the corresponding coordinates; $\Delta x(k), \Delta z(k)$ are the measurements errors), then expression (22) for parity control takes the form

$$r(k) = v_3 \Delta x(k) + v_4 \Delta z(k). \tag{23}$$

From the set (21) we select two vectors $\mathbf{v}^{(1)} = \{v_i^{(1)}\}$, $\mathbf{v}^{(2)} = \{v_i^{(2)}\}$ $(i = \overline{1, 4})$ satisfying the condition $v_3^{(1)} v_4^{(2)} - v_3^{(2)} v_4^{(1)} \neq 0$ (or $v_1^{(1)} v_2^{(2)} - v_1^{(2)} v_2^{(1)} \neq 0$, that is equivalent). Then the system of equations

$$r^{(1)}(k) = v_3^{(1)} \Delta x(k) + v_4^{(1)} \Delta z(k),$$
$$r^{(2)}(k) = v_3^{(2)} \Delta x(k) + v_4^{(2)} \Delta z(k) \tag{24}$$

is solvable with respect to $\Delta x(k)$, $\Delta z(k)$. Fulfillment, for example, of condition

$$\min \left(\left| \Delta x(k) \right|, \left| \Delta z(k) \right| \right) > N_S, \tag{25}$$

in which N_S is a priori given admissible level of devise error suggests the failure of Sun sensor. Otherwise, the optical system is in state of operability. There could be another different from (25) condition stating the sensor failure.

Thus, on the next step of analyzing Sun sensor operability one needs to form and solve the system of equations (24). For this purpose it is necessary to apply the information on coordinates x and z of Sun position on the plane of images of optical system as well as data on projections of vector of SC absolute angular rate on the axis of related system of coordinates referring to the previous and current steps of measurements. Further, for each vector $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ of the set (21) we calculate their corresponding values $r^{(1)}$ and $r^{(2)}$ (formula (22)).

The operation of identification algorithm of sensor failure will be estimated by characterizing the direction at Sun in the coordinate system $x_S y_S z_S$ by the angles ξ and η. Counting the angle ξ of the positive direction of the axis y_S, the connection of these angles with the unit vector

$$\mathbf{S} = [s_x \ s_y \ s_z]^{\mathrm{T}} = \left[\frac{x}{q} \ \frac{F}{q} \ \frac{z}{q} \right]^{\mathrm{T}}, \quad q = \sqrt{x^2 + F^2 + z^2}$$

of the mentioned direction is determined by the relations

$$s_x = -\sin \xi \cos \eta, \ s_y = \cos \xi \cos \eta, \ s_z = \sin \eta.$$

We select the following values of vectors $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$:

$$\mathbf{v}^{(1)} = [1 \ 1 \ s - c \ -(s + c)]^{\mathrm{T}}, \ \mathbf{v}^{(2)} = [1 \ 3 \ 3s - c \ -(s + 3c)]^{\mathrm{T}}.$$

In mathematical simulation the use was made of fragments of recording of vectors $\mathbf{S}$ and $\boldsymbol{\omega}$ with the step $h = 1$ s referring to SC motion in mode of triaxial orbital attitude on the 500 seconds' time interval. One can make judgment about the features of operation of failure identification algorithm by data (their dimensionality being degree) presented in Table 2. For each of measuring channels of Sun position there are assigned two columns. The first contains the accurate values of displacements $\Delta\xi$ and $\Delta\eta$ in Sun sensor readings introduced for investigating identification algorithm operability; the second contains the estimates $\Delta\hat{\xi}$ and $\Delta\hat{\eta}$ of these displacements quantities calculated by identification algorithm. The error was estimated every second of SC motion on the above time interval.

Table 2

Variant number	x-channel		z-channel	
	$\Delta\xi$	$\Delta\hat{\xi}$	$\Delta\eta$	$\Delta\hat{\eta}$
1	0	0.012	0	0.054
2	1.1	1.114	0	0.059
3	0	0.013	-0.8	-0.843
4	-1.0	-0.988	1.5	1.554
5	0.8	0.813	-1.2	-1.143

The goal of the simulation first variant is to estimate the level of systematic errors of algorithm which are accounted for by approximating the solution of the system of equations (18) describing the Sun

motion on the plane $x_S z_S$ by solution (11), (20) of equations (19) while stabilizing the mode of SC triaxial orbital attitude. The next variants present typical values of displacements estimates $\Delta\hat{\xi}$ and $\Delta\hat{\eta}$ for different combinations of qualities and signs of "accurate" displacements. Table 2 contains the worst (the largest by module) values of errors $\Delta\hat{\xi}$ and $\Delta\hat{\eta}$ on the 500 seconds' time interval.

As the analysis of simulation results implies, on the used fragment of recordings of vectors **S** and **ω** the systematic error of identification algorithm of Sun sensor failures does not exceed 4 ang. min. Approximately with the same error the angles ξ and η are estimated in other variants of simulation. With the threshold $N = 1$ degree and condition (25) in simulation variants 2, 4, 5 the algorithm diagnoses Sun sensor failure.

The measurements x and z imitated while simulation did not contain random noises. Their presence naturally increases the values of errors $\Delta\hat{\xi}$ and $\Delta\hat{\eta}$. However, this source of errors is nor related directly to the identification algorithm since it characterizes the uncertainty degree of initial information applied in the identification algorithm. A priori information on the level of random noises is taken into account when assigning the quantity of threshold N.

Conclusion

If the fail-safe unit of ARS contains five sensors (five measuring channels), then a single failure in one of them is identified by the above algorithm. These results in a failed device being eliminated from the measuring system of SC angular rate (or with the known model of failure, for example, a permanent displacement this information can be applied to determining the next failures [6]). In this paper with the unknown model of failure the readings of failed ARS are substituted by "virtual" measurements formed by data on a vector of angular rate **ω** from the previous measurement step and so on. Therewith there are preserved five measuring channels and identification algorithm of failures of actually operating sensors. The identification procedure terminates on detecting the failure of one of three remaining ARS.

Although the presence of systematic errors of the form (16) narrows the possibilities of the proposed identification procedure of sensing elements failures, the above technique of arranging "virtual" measurements allows one on a single algorithmic base to expand substantially the possibilities of internal reservation when constructing fail-safe measuring systems extending the failure identification procedure to the case of three simultaneously operating sensors.

When solving the identification problem of Sun sensor failures of importance were the equations (18) of Sun motion on the image plane of optical system and their discrete approximation in the form of relations (19), (20). As simulation results imply, the thereby constructed algorithm of failures localization effectively solves the stated problem.

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Absolute Interval Stability of Indirect Regulating Systems of Neutral Type

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ABSTRACT

The Lyapunov direct method with the Lyapunov–Krasovskiy functional is under investigation. We proved sufficient criteria of stability, namely, absolute by nonlinearity and interval by parameters. Criteria of stability and estimates of convergence of solutions are given in the form of constructive algebraic inequalities.

Key words: Lyapunov direct method, Lyapunov–Krasovskiy functional, sufficient criteria of stability, absolute and interval stability, estimates of convergence of solutions, constructive algebraic inequalities.

In the present article we obtained sufficient conditions of absolute interval stability of nonlinear control systems, described by differential-difference equations of neutral type with one deviation of argument. The Lyapunov–Krasovskiy method of functionals is under investigation. Such kind of problems appear because of necessity of providing global asymptotic stability of zero solution of control systems of special ("sector") type [1–3]. Initially we consider systems, described by differential equations. Then, investigations were done for discrete and differential-difference equations [4–6]. As a rule, parameters of systems are exactly unknown. They take on their values from certain in advance given intervals. Therefore, the direction of investigation of robust or interval stability appeared [7]. Survey of investigations of problems of absolute stability is contained in publications [8, 9].

1. Absolute interval stability

Let us consider a system of indirect control, which is described by differential-difference equations of neutral type

$$\frac{d}{dt}[x(t) - Dx(t - \tau)] = (A + \Delta A)\, x(t) + (B + \Delta B)\, x(t - \tau) + bf(\sigma(t)),$$

$$\dot{\sigma}(t) = c^{\mathrm{T}} x(t) - \rho f(\sigma(t)), \ \ \rho > 0, \ \ x(t) \in R^n, \ \ \sigma(t) \in R^1, \ \ t \geq 0, \tag{1}$$

with constant quadratic matrixes A, B, $D \in R^{n \times n}$ and vectors b, $c \in R^n$. The solution of a system implies a pair of piecewise-continuously-differentiable functions $(x(t), \sigma(t))$ with initial conditions $x(t) = \varphi(t)$, $\dot{x}(t) = \phi(t)$, $\sigma(t) = \varphi_1(t)$, $\dot{\sigma}(t) = \phi_1(t)$, which identically hold the system (1). Matrix $|D|$ satisfies "stability conditions", i.e., $|D| < 1$, is fulfilled, and constant quadratic matrixes ΔA, ΔB can take on their values from certain in advance fixed intervals

$$\Delta A = \{\Delta a_{ij}\}, \ \ \Delta B = \{\Delta b_{ij}\}, \ \ \left|\Delta a_{ij}\right| \leq \alpha_{ij}, \ \ \left|\Delta b_{ij}\right| \leq \beta_{ij}, \ \ i, j = \overline{i, n}. \tag{2}$$

The systems of such type got the name of interval indirect regulating systems. The nonlinear function $f(\sigma)$ lies in the given sector, i.e., holds the conditions

$$k_1 \sigma^2 < f(\sigma)\sigma < k_2 \sigma^2, \ \ k_2 > k_1 > 0. \tag{3}$$

Let us introduce the following denotations:

$$|A| = \{\lambda_{\max}(A^{\mathrm{T}} A)\}^{1/2}, \ \ |x(t)| = \left\{\sum_{s=1}^{n} x_i^2(t)\right\}^{1/2}, \ \ \|x(t)\|_{\tau,\varsigma} = \left\{\int_{-\tau}^{0} e^{-\varsigma(t-s)} |x(s)|^2 \, ds\right\}^{1/2},$$

$$\|\Delta A\| = \max_{\Delta a_{ij}}\{|\Delta A|\}, \ \ \|\Delta B\| = \max_{\Delta a_{ij}}\{|\Delta B|\},$$

$\lambda_{\max}(\cdot)$, $\lambda_{\min}(\cdot)$ are extremal eigenvalues of the corresponding symmetric positively-defined matrixes.

Let us consider initially a system without interval perturbations

$$\begin{cases} \dfrac{d}{dt}[x(t) - Dx(t - \tau)] = Ax(t) + Bx(t - \tau) + bf(\sigma(t)), \\[2mm] \dot{\sigma}(t) = c^{\mathrm{T}} x(t) - \rho f(\sigma(t)). \end{cases} \tag{4}$$

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Definition 1. The system (4) is called absolutely stable, if its zero solution $x(t) \equiv 0$, $\sigma(t) \equiv 0$ is globally asymptotically stable under arbitrary function $f(\sigma)$, satisfying the conditions (3).

Definition 2. The system (4) is called absolutely interval stable, if the system (1) is absolutely stable for arbitrary matrixes ΔA, ΔB, satisfying the conditions (2).

On investigation of absolute stability of systems of neutral type the Lyapunov–Krasovskiy functional of the following form [4, 5] is used sufficiently frequently

$$V[x(t), \sigma(t)] = (x(t) - Dx(t - \tau))^{\mathrm{T}} H(x(t) - Dx(t - \tau)) +$$

$$+ \gamma \int_{t-\tau}^{t} x^{\mathrm{T}}(s) Hx(s) \, ds + \beta \int_{0}^{\sigma(t)} f(\sigma) \, d\sigma, \quad \gamma > 0, \ \beta > 0. \tag{5}$$

The first quadratic form makes it possible to compute easily total derivative of the functional in virtue of the system, however it is possible to obtain conditions of absolute stability only in integral metric.

In the present article we investigate absolute interval stability, i.e., stability of zero solution of the system (1) under perturbations (2). Here one uses the functional with exponential multiplier

$$V[x(t), \sigma(t)] = (x(t) - Dx(t - \tau))^{\mathrm{T}} H(x(t) - Dx(t - \tau)) +$$

$$+ \int_{t-\tau}^{t} e^{-\varsigma(t-s)} x^{\mathrm{T}}(s) Gx(s) \, ds + \beta \int_{0}^{\sigma(t)} f(\sigma) \, d\sigma, \quad \varsigma > 0, \tag{6}$$

and two positively-defined matrixes G, H. Exponential multiplier makes it possible to obtain not only stability conditions, but also compute estimates of convergence of solutions of the system (1).

Preliminarily we obtain conditions of absolute system stability without interval perturbations (4). Let us denote

$$M[H] = \begin{bmatrix} H & HD \\ D^{\mathrm{T}}H & D^{\mathrm{T}}HD \end{bmatrix}, \tag{7}$$

$$S_1[G, H, \beta, \varsigma] = \begin{bmatrix} -A^{\mathrm{T}}H - HA - G & -HB + D^{\mathrm{T}}HA & -\left[Hb + \dfrac{1}{2}\beta c\right] \\ -B^{\mathrm{T}}H + A^{\mathrm{T}}HD & B^{\mathrm{T}}HD + D^{\mathrm{T}}HB + e^{-\varsigma\tau}G & D^{\mathrm{T}}Hb \\ -\left[Hb + \dfrac{1}{2}\beta c\right]^{\mathrm{T}} & b^{\mathrm{T}}HD & \beta\rho \end{bmatrix}. \tag{8}$$

Theorem 1. Let positively-defined matrixes G, H and parameters $\beta > 0$, $\varsigma > 0$ exist such, that the matrix $S_1[G, H, \beta, \varsigma]$ is also positively-defined. Then the system (4) is absolutely stable in metric $\|x(t)\|_{\tau,\varsigma}$, $|\sigma(t)|$.

Proof. Taking into account constraints (3), superimposed on the function $f(\sigma)$, we obtain that for the Lyapunov–Krasovskiy functional (6) the following bilateral inequalities take place

$$\lambda_{\min}(G) \|x(t)\|_{\tau,\varsigma}^2 + \frac{1}{2}\beta k_1 \sigma^2(t) \leq V[x(t), \sigma(t)] \leq$$

$$\leq \lambda_{\max}(M[H])[|x(t)|^2 + |x(t-\tau)|^2] + \lambda_{\max}(G)\|x(t)\|_{\tau,\varsigma}^2 + \frac{1}{2}\beta k_2 \sigma^2(t). \tag{9}$$

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If we compute total derivative of the functional (6) in virtue of the system of non-interval perturbations (4) we obtain the following:

$$\frac{d}{dt}V[x(t),\,\sigma(t)] = [Ax(t) + Bx(t-\tau) + bf(\sigma(t))]^{\mathrm{T}}\, H(x(t) - Dx(t-\tau)) +$$

$$+ (x(t) - Dx(t-\tau))^{\mathrm{T}} H[Ax(t) + Bx(t-\tau) + bf(\sigma(t))] +$$

$$+ x^{\mathrm{T}}(t)\, Gx(t) - e^{-\varsigma\tau} x^{\mathrm{T}}(t-\tau)\, Gx(t-\tau) +$$

$$+ \beta f(\sigma(t))[c^{\mathrm{T}} x(t) - \rho f(\sigma(t))] - \varsigma \int_{t-\tau}^{t} e^{-\varsigma(t-s)} x^{\mathrm{T}}(s)\, Gx(s)\, ds.$$

Using the vector-matrix form we rewrite the obtained expression:

$$\frac{d}{dt}V[x(t),\,\sigma(t)] = -(x^{\mathrm{T}}(t),\, x^{\mathrm{T}}(t-\tau),\, f(\sigma(t))) \times$$

$$\times \begin{bmatrix} -A^{\mathrm{T}}H - HA - G & -HB + D^{\mathrm{T}}HA & -\left[Hb + \dfrac{1}{2}\beta c\right] \\[2mm] -B^{\mathrm{T}}H + A^{\mathrm{T}}HD & B^{\mathrm{T}}HD + D^{\mathrm{T}}HB + e^{-\varsigma\tau}G & D^{\mathrm{T}}Hb \\[2mm] -\left[Hb + \dfrac{1}{2}\beta c\right]^{\mathrm{T}} & b^{\mathrm{T}}HD & \beta\rho \end{bmatrix} \begin{pmatrix} x(t) \\[2mm] x(t-\tau) \\[2mm] f(\sigma(t)) \end{pmatrix} -$$

$$- \varsigma \int_{t-\tau}^{t} e^{-\varsigma(t-s)} x^{\mathrm{T}}(s)\, Gx(s)\, ds.$$

If the matrix $S_1[G,\,H,\,\beta,\,\varsigma]$ is positively-defined, then

$$\frac{d}{dt}V[x(t),\,\sigma(t)] \leq -\lambda_{\min}(S_1[G,\,H,\,\beta,\,\varsigma])(|x(t)|^2 +$$

$$+ |x(t-\tau)|^2 + |f(\sigma(t))|^2) - \varsigma \int_{t-\tau}^{t} e^{-\varsigma(t-s)} x^{\mathrm{T}}(s)\, Gx(s)\, ds.$$

Thus we have the system of inequalities

$$\lambda_{\min}(G)\|x(t)\|_{\tau,\varsigma}^2 + \frac{1}{2}\beta k_1 \sigma^2(t) \leq V[x(t),\,\sigma(t)] \leq$$

$$\leq \lambda_{\max}(M[H])[|x(t)|^2 + |x(t-\tau)|^2] + \lambda_{\max}(G)\|x(t)\|_{\tau,\varsigma}^2 + \frac{1}{2}\beta k_2 \sigma^2(t),$$

$$\frac{d}{dt}V[x(t),\,\sigma(t)] \leq -\lambda_{\min}(S_1[G,\,H,\,\beta,\,\varsigma])[|x(t)|^2 +$$

$$+ |x(t-\tau)|^2] - \varsigma\lambda_{\min}(G)\|x(t)\|_{\tau,\varsigma}^2 - \lambda_{\min}(S_1[G,\,H,\,\beta,\,\varsigma]) k_1^2 \sigma^2(t).$$

As it follows from the Krasovskiy theorem [10, p. 145], the system is absolutely stable in the metric $\|x(t)\|_{\tau,\varsigma}$, $|\sigma(t)|$.

Further we obtain conditions of absolutely interval stability of the system (4). Let write preliminarily the following auxiliary result.

Lemma. For arbitrary matrixes L_1, L_2 of the vectors $x(t)$, $x(t-\tau)$ and the scalar ξ the following inequality takes place:

$$x^{\mathrm{T}}(t)L_1^{\mathrm{T}}L_2 x(t-\tau) + x^{\mathrm{T}}(t-\tau)L_2^{\mathrm{T}}L_1 x(t) \le$$

$$\le \xi^2 x^{\mathrm{T}}(t)L_1^{\mathrm{T}}L_1 x(t) + \frac{1}{\xi^2}x^{\mathrm{T}}(t-\tau)L_2^{\mathrm{T}}L_2 x(t-\tau). \tag{10}$$

Proof. If we open evident expression for arbitrary matrixes L_1, L_2 of the vectors $x(t)$, $x(t-\tau)$ and the scalar ξ, we obtain the following:

$$\left[\xi L_1 x(t) - \frac{1}{\xi}L_2 x(t-\tau)\right]^{\mathrm{T}}\left[\xi L_1 x(t) - \frac{1}{\xi}L_2 x(t-\tau)\right] = \xi^2 x^{\mathrm{T}}(t)L_1^{\mathrm{T}}L_1 x(t) +$$

$$+ \frac{1}{\xi^2}x^{\mathrm{T}}(t-\tau)L_2^{\mathrm{T}}L_2 x(t-\tau) - x^{\mathrm{T}}(t)L_1^{\mathrm{T}}L_2 x(t-\tau) - x^{\mathrm{T}}(t-\tau)L_2^{\mathrm{T}}L_1 x(t) \ge 0.$$

We obtain from this

$$x^{\mathrm{T}}(t)L_1^{\mathrm{T}}L_2 x(t-\tau) + x^{\mathrm{T}}(t-\tau)L_2^{\mathrm{T}}L_1 x(t) \le$$

$$\le \xi^2 x^{\mathrm{T}}(t)L_1^{\mathrm{T}}L_1 x(t) + \frac{1}{\xi^2}x^{\mathrm{T}}(t-\tau)L_2^{\mathrm{T}}L_2 x(t-\tau),$$

i.e., the inequality (10).

Let us denote

$$S_2[G, H] = \begin{bmatrix} \Delta A^{\mathrm{T}}H + H\Delta A & H\Delta B - \Delta A^{\mathrm{T}}HD & \theta \\ \Delta B^{\mathrm{T}}H - D^{\mathrm{T}}H\Delta A & -\Delta B^{\mathrm{T}}HD - D^{\mathrm{T}}H\Delta B & \theta \\ \theta^{\mathrm{T}} & \theta^{\mathrm{T}} & 0 \end{bmatrix},$$

where θ is zero vector.

Theorem 2. Let positively-defined matrixes G, H and parameters $\beta > 0$, $\varsigma > 0$ exist, for which the matrix $S_1[G, H, \beta, \varsigma]$ be positively-defined, and for the given $0 < \xi < 1$ the following inequalities hold:

$$\|\Delta A\| < \frac{\xi\lambda_{\min}(S_1[G, H, \beta, \varsigma])}{|HD|}\left[\sqrt{\left(\frac{\xi\lambda_{\max}(H)}{|HD|}\right)^2 - (1-\xi^2)} - \frac{\xi\lambda_{\max}(H)}{|HD|}\right],$$

$$\|\Delta B\| < \frac{\xi\lambda_{\min}(S_1[G, H, \beta, \varsigma])}{\lambda_{\max}(H)}\left[\sqrt{\left(\frac{\xi|HD|}{\lambda_{\max}(H)}\right)^2 - (1-\xi^2)} - \frac{\xi|HD|}{\lambda_{\max}(H)}\right]. \tag{11}$$

Then the system (4) is absolutely interval stable in metric $\|x(t)\|_{\tau,\varsigma}$, $|\sigma(t)|$.

Proof. Let us compute total derivative of the Lyapunov–Krasovskiy functional (6) in virtue of the interval system (1)

$$\frac{d}{dt}V[x(t), \sigma(t)] = [(A + \Delta A)x(t) + (B + \Delta B)x(t-\tau) + bf(\sigma(t))]^{\mathrm{T}}H(x(t) - Dx(t-\tau)) +$$

$$+ (x(t) - Dx(t - \tau))^{\mathrm{T}} H[(A + \Delta A) x(t) + (B + \Delta B) x(t - \tau) + bf(\sigma(t))] +$$

$$+ x^{\mathrm{T}}(t) Gx(t) - e^{-\varsigma\tau} x^{\mathrm{T}}(t - \tau) Gx(t - \tau) +$$

$$+ \beta f(\sigma(t))[c^{\mathrm{T}} x(t) - \rho f(\sigma(t))] - \varsigma \int\limits_{t-\tau}^{t} e^{-\varsigma(t-s)} x^{\mathrm{T}}(s) Gx(s)\, ds.$$

Using the vector-matrix form of notation we rewrite the given expression in the form

$$\frac{d}{dt} V[x(t), \sigma(t)] = -(x^{\mathrm{T}}(t), x^{\mathrm{T}}(t - \tau), f(\sigma(t))) S_1[G, H, \beta, \varsigma] \times$$

$$\times (x^{\mathrm{T}}(t), x^{\mathrm{T}}(t - \tau), f(\sigma(t)))^{\mathrm{T}} + (x^{\mathrm{T}}(t), x^{\mathrm{T}}(t - \tau), f(\sigma(t))) \times$$

$$\times S_2[G, H](x^{\mathrm{T}}(t), x^{\mathrm{T}}(t - \tau), f(\sigma(t)))^{\mathrm{T}} - \varsigma \int\limits_{t-\tau}^{t} e^{-\varsigma(t-s)} x^{\mathrm{T}}(s) Gx(s)\, ds.$$

Let us expose the second quadratic form

$$(x^{\mathrm{T}}(t), x^{\mathrm{T}}(t - \tau), f(\sigma(t))) S_2[G, H](x^{\mathrm{T}}(t), x^{\mathrm{T}}(t - \tau), f(\sigma(t)))^{\mathrm{T}} =$$

$$= x^{\mathrm{T}}(t)[\Delta A^{\mathrm{T}} H + H\Delta A] x(t) + x^{\mathrm{T}}(t - \tau)[\Delta B^{\mathrm{T}} H - D^{\mathrm{T}} H\Delta A] x(t) +$$

$$+ x^{\mathrm{T}}(t)[H\Delta B - \Delta A^{\mathrm{T}} HD] x(t - \tau) + x^{\mathrm{T}}(t - \tau)[-\Delta B^{\mathrm{T}} HD - D^{\mathrm{T}} H\Delta B] x(t - \tau).$$

We consider every of addends separately. Using results of Lemma we obtain the following.
1. For the first addend

$$x^{\mathrm{T}}(t)[\Delta A^{\mathrm{T}} H + H\Delta A] x(t) \le 2\lambda_{\max}(H)\big|\Delta A\big|\big|x(t)\big|^2.$$

2. For the second and third one

$$x^{\mathrm{T}}(t - \tau)[\Delta B^{\mathrm{T}} H - D^{\mathrm{T}} H\Delta A] x(t) + x^{\mathrm{T}}(t)[H\Delta B - \Delta A^{\mathrm{T}} HD] x(t - \tau) =$$

$$= \{x^{\mathrm{T}}(t - \tau) \Delta B^{\mathrm{T}} Hx^{\mathrm{T}}(t) + x^{\mathrm{T}}(t) H\Delta Bx(t - \tau)\} -$$

$$- \{x^{\mathrm{T}}(t - \tau) D^{\mathrm{T}} H\Delta Ax(t) + x^{\mathrm{T}}(t) \Delta A^{\mathrm{T}} HDx(t - \tau)\} \le$$

$$\le \left\{\xi_1^2 x^{\mathrm{T}}(t - \tau) \Delta B^{\mathrm{T}} \Delta Bx(t - \tau) + \frac{1}{\xi_1^2} x^{\mathrm{T}}(t) HHx(t)\right\} +$$

$$+ \left\{\xi_2^2 x^{\mathrm{T}}(t - \tau) D^{\mathrm{T}} HHDx(t - \tau) + \frac{1}{\xi_2^2} x^{\mathrm{T}}(t) \Delta A^{\mathrm{T}} \Delta Ax(t)\right\} \le$$

$$\le \left\{\frac{1}{\xi_1^2} \lambda_{\max}^2(H) + \frac{1}{\xi_2^2}\big|\Delta A\big|^2\right\}\big|x(t)\big|^2 + \{\xi_1^2\big|\Delta B\big|^2 + \xi_2^2\big|HD\big|^2\}\big|x(t - \tau)\big|^2.$$

3. For the fourth one

$$x^{\mathrm{T}}(t - \tau)[-\Delta B^{\mathrm{T}} HD - D^{\mathrm{T}} H\Delta B] x(t - \tau) \le 2\big|HD\big|\big|\Delta B\big|\big|x(t - \tau)\big|^2.$$

If we substitute the written expressions into estimate for total derivative, we obtain

$$\frac{d}{dt}V[x(t),\,\sigma(t)] \le$$

$$\le -\left\{\lambda_{\min}(S_1[G,H,\beta,\varsigma]) - \left[2\lambda_{\max}(H)|\Delta A| + \frac{1}{\xi_1^2}\lambda_{\max}^2(H) + \frac{1}{\xi_2^2}|\Delta A|^2\right]\right\}|x(t)|^2 -$$

$$-\{\lambda_{\min}(S_1[G,H,\beta,\varsigma]) - [2|HB||\Delta B| + \xi_1^2|\Delta B|^2 + \xi_2^2|HD|^2]\}|x(t-\tau)|^2 -$$

$$-\lambda_{\min}(S_1[G,H,\beta,\varsigma])k_1^2\sigma^2(t) - \varsigma\int\limits_{t-\tau}^{t}e^{-\varsigma(t-s)}x^{\mathrm{T}}(s)Gx(s)\,ds.$$

The condition of negative definiteness of total derivative is realization of the following relations:

$$\theta_1[\cdot] = \lambda_{\min}(S_1[G,H,\beta,\varsigma]) - 2\lambda_{\max}(H)|\Delta A| - \frac{1}{\xi_1^2}\lambda_{\max}^2(H) - \frac{1}{\xi_2^2}|\Delta A|^2 > 0,$$

$$\theta_2[\cdot] = \lambda_{\min}(S_1[G,H,\beta,\varsigma]) - 2|HD||\Delta B| - \xi_1^2|\Delta B|^2 - \xi_2^2|HD|^2 > 0.$$

Let us assume

$$\xi_1^2 = \frac{\lambda_{\max}^2(H)}{\xi^2\lambda_{\min}(S_1[G,H,\beta,\varsigma])}, \quad \xi_2^2 = \frac{\xi^2\lambda_{\min}(S_1[G,H,\beta,\varsigma])}{|HD|^2}, \quad \xi^2 < 1.$$

Then these inequalities will have the form

$$\theta_1[\cdot] = (1-\xi^2)\lambda_{\min}(S_1[G,H,\beta,\varsigma]) -$$

$$- 2\lambda_{\max}(H)|\Delta A| - \frac{|HD|^2}{\xi^2\lambda_{\min}(S_1[G,H,\beta,\varsigma])}|\Delta A|^2 > 0,$$

$$\theta_2[\cdot] = (1-\xi^2)\lambda_{\min}(S_1[G,H,\beta,\varsigma]) -$$

$$- 2|HD||\Delta B| - \frac{\lambda_{\max}^2(H)}{\xi^2\lambda_{\min}(S_1[G,H,\beta,\varsigma])}|\Delta B|^2 > 0.$$

If we solve the first inequality with respect to $|\Delta A|$, we obtain that for $0 < \xi < 1$ and

$$|\Delta A| < \frac{\xi\lambda_{\min}(S_1[G,H,\beta,\varsigma])}{|HD|}\left[\sqrt{\left(\frac{\xi\lambda_{\max}(H)}{|HD|}\right)^2 + (1-\xi^2)} - \frac{\xi\lambda_{\max}(H)}{|HD|}\right]$$

the function $\theta_1[\cdot]$ is positive. Similar for the second inequality the function $\theta_2[\cdot]$ will be positive for

$$|\Delta B| < \frac{\xi\lambda_{\min}(S_1[G,H,\beta,\varsigma])}{\lambda_{\max}(H)}\left[\sqrt{\left(\frac{\xi|HD|}{\lambda_{\max}(H)}\right)^2 + (1-\xi^2)} - \frac{\xi|HD|}{\lambda_{\max}(H)}\right].$$

So, on realization of the conditions (11) $\theta_1[\cdot] > 0$ and $\theta_2[\cdot] > 0$ will be fulfilled and

$$\frac{d}{dt}V[x(t), \sigma(t)] \le -\theta_1[\cdot]\big|x(t)\big|^2 - \theta_2[\cdot]\big|x(t-\tau)\big|^2 -$$

$$- \lambda_{\min}(S_1[G, H, \beta, \varsigma])k_1^2\sigma^2(t) - \varsigma\lambda_{\min}(G)\big\|x(t)\big\|_{\tau,\varsigma}^2 .$$

According to [10, p. 145] the system (1) will be absolutely interval stable.

2. Estimates of convergence of solutions of interval systems

In previous theorems we obtained conditions of absolute stability. At the same time for solving practical problems more important is not only the fact of stability, but computation of estimates of solution convergence. For obtaining estimates of convergence for solutions of interval system with delay we shall use again the Lyapunov–Krasovskiy functional (6).

Let us denote

$$\varphi_{11}(G, H) = \frac{\lambda_{\max}(M[H])}{\lambda_{\min}(G)}, \ \ \varphi_{12}(G) = \frac{\lambda_{\max}(G)}{\lambda_{\min}(G)}, \ \ \varphi_{13}(G) = \frac{\beta k_2}{2\lambda_{\min}(G)},$$

$$\varphi_{21}(H) = \frac{2\lambda_{\max}(M[H])}{\beta k_1}, \ \ \varphi_{22}(G) = \frac{2\lambda_{\max}(G)}{\beta k_1}, \ \ \varphi_{23} = \frac{k_2}{k_1}. \tag{12}$$

The following statement takes place.

Theorem 3. Let positively-defined matrixes G, H and scalars $\beta > 0$, $\varsigma > 0$ exist, for which the matrix $S_1[G, H, \beta, \varsigma]$ is positively-defined and conditions (11) are fulfilled. Then for the pair of solutions $(x(t), \sigma(t))$ of the interval system of neutral type (1) the following estimates of convergence take place:

$$\big\|x(t)\big\|_{\tau,\varsigma} \le [\sqrt{\varphi_{11}(G, H)}\big|x(0)\big| + \sqrt{\varphi_{11}(G, H)}\big|x(-\tau)\big| +$$

$$+ \sqrt{\varphi_{12}(G)}\big\|x(0)\big\|_{\tau,\varsigma} + \sqrt{\varphi_{13}(G)}\,]\exp\Big\{-\frac{1}{2}\chi[\cdot]t\Big\},$$

$$\big|\sigma(t)\big| \le [\sqrt{\varphi_{21}(G)}\big|x(0)\big| + \sqrt{\varphi_{21}(G)}\big|x(-\tau)\big| + \tag{13}$$

$$+ \sqrt{\varphi_{22}(G)}\big\|x(0)\big\|_{\tau,\varsigma} + \sqrt{\varphi_{23}}\,]\exp\Big\{-\frac{1}{2}\chi[\cdot]t\Big\},$$

where

$$\chi[\cdot] = \min\left\{\frac{\theta_1[\cdot]}{\lambda_{\max}(M[H])}, \frac{\theta_2[\cdot]}{\lambda_{\max}(M[H])}, \varsigma\,\frac{2\lambda_{\min}(S_1[G, H, \beta, \varsigma])k_1^2}{\beta k_2}\right\}. \tag{14}$$

Proof. For obtaining the convergence conditions (13), (14) we use the Lyapunov–Krasovskiy functional of (6) type. As it was shown on proof of Theorem 2, it holds bilateral estimates

$$\int_{t-\tau}^{t} e^{-\varsigma(t-s)}x^{\mathrm{T}}(s)Gx(s)\,ds + \frac{1}{2}\beta k_1\sigma^2(t) \le V[x(t), \sigma(t)] \le \lambda_{\max}(M[H])\big|x(t)\big|^2 + \lambda_{\max}(M[H])\big|x(t-\tau)\big|^2 +$$

$$+ \int_{t-\tau}^{t} e^{-\varsigma(t-s)}x^{\mathrm{T}}(s)Gx(s)\,ds + \frac{1}{2}\beta k_2\sigma^2(t); \tag{15}$$

$$\frac{d}{dt}V[x(t),\sigma(t)] \le -\theta_1[\cdot]\big|x(t)\big|^2 - \theta_2[\cdot]\big|x(t-\tau)\big|^2 -$$

$$-\lambda_{\min}(S_1[G,H,\beta,\varsigma])k_1^2\sigma^2(t) - \varsigma \int\limits_{t-\tau}^{t} e^{-\varsigma(t-s)}x^{\mathrm{T}}(s)Gx(s)\,ds. \tag{16}$$

Let us write the right-hand part of the inequality (15) as

$$-\lambda_{\max}(M[H])\big|x(t)\big|^2 - \lambda_{\max}(M[H])\big|x(t-\tau)\big|^2 -$$

$$-\int\limits_{t-\tau}^{t} e^{-\varsigma(t-s)}x^{\mathrm{T}}(s)Gx(s)\,ds - \frac{1}{2}\beta k_2\sigma^2(t) \le -V[x(t),\sigma(t)]. \tag{17}$$

We consider the following cases.

1. Let us write the inequality (17) as

$$-\big|x(t)\big|^2 \le \frac{1}{\lambda_{\max}(M[H])}\times$$

$$\times\left\{-V[x(t),\sigma(t)] + \lambda_{\max}(M[H])\big|x(t-\tau)\big|^2 + \int\limits_{t-\tau}^{t} e^{-\varsigma(t-s)}x^{\mathrm{T}}(s)Gx(s)\,ds + \frac{1}{2}\beta k_2\sigma^2(t)\right\}.$$

If we substitute the obtained expression into (16), we have

$$\frac{d}{dt}V[x(t),\sigma(t)] \le -\frac{\theta_1[\cdot]}{\lambda_{\max}(M[H])}V[x(t),\sigma(t)] - \{\theta_2[\cdot] - \theta_1[\cdot]\}\big|x(t-\tau)\big|^2 -$$

$$-\left\{\lambda_{\min}(S_1[G,H,\beta,\varsigma])k_1^2 - \frac{\theta_1[\cdot]}{\lambda_{\max}(M[H])}\frac{1}{2}\beta k_2\right\}\sigma^2(t) -$$

$$-\left\{\varsigma - \frac{\theta_1[\cdot]}{\lambda_{\max}(M[H])}\right\}\int\limits_{t-\tau}^{t} e^{-\varsigma(t-s)}x^{\mathrm{T}}(s)Gx(s)\,ds.$$

If parameters of the system and the functional are such that

$$\theta_2[\cdot] - \theta_1[\cdot] \ge 0, \quad \lambda_{\min}(S_1[G,H,\beta,\varsigma])k_1^2 - \frac{\theta_1[\cdot]\beta k_2}{2\lambda_{\max}(M[H])} \ge 0, \quad \varsigma - \frac{\theta_1[\cdot]}{\lambda_{\max}(M[H])} \ge 0,$$

then

$$\frac{d}{dt}V[x(t),\sigma(t)] \le -\frac{\theta_1[\cdot]}{\lambda_{\max}(M[H])}V[x(t),\sigma(t)].$$

If we solve the obtained inequality, we have

$$V[x(t),\sigma(t)] \le V[x(0),\sigma(0)]\exp\left\{-\frac{\theta_1[\cdot]}{\lambda_{\max}(M[H])}\right\}t, \quad t \ge 0.$$

2. Let us write the inequality (17) as

$$-\big|x(t-\tau)\big|^2 \le \frac{1}{\lambda_{\max}(M[H])}\times$$

$$\times \left\{ -V[x(t),\sigma(t)] + \lambda_{\max}(M[H])\big|x(t)\big|^2 + \int_{t-\tau}^{t} e^{-\varsigma(t-s)} x^{\mathrm{T}}(s)\,Gx(s)\,ds + \frac{1}{2}\beta k_2 \sigma^2(t) \right\}.$$

If we substitute the obtained expression into (16), we obtain

$$\frac{d}{dt}V[x(t),\,\sigma(t)] \le -\frac{\theta_2[\cdot]}{\lambda_{\max}(M[H])} V[x(t),\,\sigma(t)] - \{\theta_1[\cdot] - \theta_2[\cdot]\}\big|x(t)\big|^2 -$$

$$- \left\{ \lambda_{\min}(S_1[G,H,\beta,\varsigma])\,k_1^2 - \frac{\theta_2[\cdot]}{\lambda_{\max}(M[H])}\frac{1}{2}\beta k_2 \right\}\sigma^2(t) -$$

$$- \left\{ \varsigma - \frac{\theta_2[\cdot]}{\lambda_{\max}(M[H])} \right\} \int_{t-\tau}^{t} e^{-\varsigma(t-s)} x^{\mathrm{T}}(s)\,Gx(s)\,ds.$$

If parameters of the system and the functional are such that

$$\theta_1[\cdot] - \theta_2[\cdot] \ge 0,\ \ \lambda_{\min}(S_1[G,H,\beta,\varsigma])\,k_1^2 - \frac{\theta_2[\cdot]\beta k_2}{2\lambda_{\max}(M[H])} \ge 0,\ \varsigma - \frac{\theta_2[\cdot]}{\lambda_{\max}(M[H])} \ge 0,$$

then

$$\frac{d}{dt}V[x(t),\,\sigma(t)] \le -\frac{\theta_2[\cdot]}{\lambda_{\max}(M[H])} V[x(t),\,\sigma(t)].$$

Solving the obtained inequality we get

$$V[x(t),\,\sigma(t)] \le V[x(0),\,\sigma(0)]\exp\left\{ -\frac{\theta_2[\cdot]}{\lambda_{\max}(M[H])} \right\}t,\ \ t \ge 0.$$

3. Let us write the inequality (17) as

$$- \int_{t-\tau}^{t} e^{-\varsigma(t-s)} x^{\mathrm{T}}(s)\,Gx(s)\,ds \le -V[x(t),\,\sigma(t)] +$$

$$+ \lambda_{\max}(M[H])\big|x(t)\big|^2 + \lambda_{\max}(M[H])\big|x(t-\tau)\big|^2 + \frac{1}{2}\beta k_2 \sigma^2(t)$$

and again substitute into (16). We obtain

$$\frac{d}{dt}V[x(t),\,\sigma(t)] \le -\varsigma V[x(t),\,\sigma(t)] - \{\theta_1[\cdot] - \varsigma\lambda_{\max}(M[H])\}\big|x(t)\big|^2 -$$

$$- \{\theta_2[\cdot] - \varsigma\lambda_{\max}(M[H])\}\big|x(t-\tau)\big|^2 - \left\{ \lambda_{\min}(S_1[G,H,\beta,\varsigma])\,k_1^2 - \frac{1}{2}\varsigma\beta k_1 \right\}\sigma^2(t).$$

If the following relation will hold

$$\theta_1[\cdot] - \varsigma\lambda_{\max}(M[H]) \ge 0,\ \ \theta_2[\cdot] - \varsigma\lambda_{\max}(M[H]) \ge 0,$$

$$\lambda_{\min}(S_1[G,H,\beta,\varsigma]) - \frac{1}{2}\varsigma\beta k_1 \ge 0,$$

then

$$\frac{d}{dt}V[x(t),\,\sigma(t)] \le -\varsigma V[x(t),\,\sigma(t)].$$

If we integrate this inequality, we get

$$V[x(t), \sigma(t)] \leq V[x(0), \sigma(0)] \exp\{-\varsigma t\}.$$

4. Let us write the inequality (17) as

$$-\sigma^2(t) \leq \frac{2}{\beta k_2} \left\{ -V[x(t), \sigma(t)] + \lambda_{\max}(M[H]) |x(t)|^2 + \lambda_{\max}(M[H]) |x(t-\tau)|^2 + \right.$$

$$\left. + \int_{t-\tau}^{t} e^{-\varsigma(t-s)} x^{\mathrm{T}}(s) Gx(s) ds + \frac{1}{2} \beta k_2 \sigma^2(t) \right\}$$

and substitute into (16). We obtain

$$\frac{d}{dt} V[x(t), \sigma(t)] \leq -\frac{2\lambda_{\min}(S_1[G, H, \beta, \varsigma]) k_1^2}{\beta k_2} V[x(t), \sigma(t)] -$$

$$- \left\{ \theta_1[\cdot] - \frac{2\lambda_{\min}(S_1[G, H, \beta, \varsigma]) k_1^2}{\beta k_2} \right\} |x(t)|^2 -$$

$$- \left\{ \theta_2[\cdot] - \frac{2\lambda_{\min}(S_1[G, H, \beta, \varsigma]) k_1^2}{\beta k_2} \right\} |x(t-\tau)|^2 -$$

$$- \left\{ \varsigma - \frac{2\lambda_{\min}(S_1[G, H, \beta, \varsigma]) k_1^2}{\beta k_2} \right\} \int_{t-\tau}^{t} e^{-\varsigma(t-s)} x^{\mathrm{T}}(s) Gx(s) ds.$$

If the following relation realizes

$$\theta_1[\cdot] - \frac{2\lambda_{\min}(S_1[G, H, \beta, \varsigma]) k_1^2}{\beta k_2} \geq 0, \quad \theta_2[\cdot] - \frac{2\lambda_{\min}(S_1[G, H, \beta, \varsigma]) k_1^2}{\beta k_2} \geq 0,$$

$$\varsigma - \frac{2\lambda_{\min}(S_1[G, H, \beta, \varsigma]) k_1^2}{\beta k_2} \geq 0,$$

then

$$\frac{d}{dt} V[x(t), \sigma(t)] \leq -\frac{2\lambda_{\min}(S_1[G, H, \beta, \varsigma]) k_1^2}{\beta k_2} V[x(t), \sigma(t)].$$

If we integrate this inequality we obtain

$$V[x(t), \sigma(t)] \leq V[x(0), \sigma(0)] \exp\left\{ -\frac{2\lambda_{\min}(S_1[G, H, \beta, \varsigma]) k_1^2}{\beta k_2} t \right\}.$$

If we unite the obtained inequalities, we have

$$V[x(t), \sigma(t)] \leq V[x(0), \sigma(0)] \exp\{-\chi[\cdot] t\}, \tag{18}$$

where

$$\chi[\cdot] = \min\left\{ \frac{\theta_1[\cdot]}{\lambda_{\max}(M[H])}, \frac{\theta_2[\cdot]}{\lambda_{\max}(M[H])}, \frac{2\lambda_{\min}(S_1[G, H, \beta, \varsigma]) k_1^2}{\beta k_2} \right\}.$$

We use again bilateral inequality of the functional $V[x(t), \sigma(t)]$ and the dependency (18). We write

$$\lambda_{\min}(G)\|x(t)\|_{\tau,\varsigma}^2 + \frac{1}{2}k_1\sigma^2(t) \le V[x(t), \sigma(t)] \le V[x(0), \sigma(0)]e^{-\chi[\cdot]t} \le$$

$$\le \left\{ \lambda_{\max}(M[H])\big|x(0)\big|^2 + \lambda_{\max}(M[H])\big|x(-\tau)\big|^2 + \right.$$

$$\left. + \lambda_{\max}(G)\|x(0)\|_{\tau,\varsigma}^2 + \frac{1}{2}k_2\sigma^2(0) \right\}e^{-\chi[\cdot]t}.$$

From this

$$\|x(t)\|_{\tau,\varsigma}^2 \le \left[\frac{\lambda_{\max}(M[H])}{\lambda_{\min}(G)}\big|x(0)\big|^2 + \frac{\lambda_{\max}(M[H])}{\lambda_{\min}(G)}\big|x(-\tau)\big|^2 + \right.$$

$$\left. + \frac{\lambda_{\max}(G)}{\lambda_{\min}(G)}\|x(0)\|_{\tau,\varsigma}^2 + \frac{k_2}{2\lambda_{\min}(G)}\sigma^2(0) \right]e^{-\chi[\cdot]t}.$$

Using denotations $\varphi_{11}(G, H)$, $\varphi_{12}(G)$, $\varphi_{13}(G)$, we obtain the following estimates of convergence:

$$\|x(t)\|_{\tau,\varsigma} \le [\sqrt{\varphi_{11}(G,H)}\big|x(0)\big| + \sqrt{\varphi_{11}(G,H)}\big|x(-\tau)\big| +$$

$$+ \sqrt{\varphi_{12}(G)}\|x(0)\|_{\tau,\varsigma} + \sqrt{\varphi_{13}(G)}]\exp\left\{-\frac{1}{2}\chi[\cdot]t\right\}.$$

We write similar

$$\big|\sigma(t)\big| \le [\sqrt{\varphi_{21}(H)}\big|x(0)\big| + \sqrt{\varphi_{21}(H)}\big|x(-\tau)\big| + \sqrt{\varphi_{22}(G)}\|x(0)\|_{\tau,\varsigma} + \sqrt{\varphi_{23}}]\exp\left\{-\frac{1}{2}\chi[\cdot]t\right\}.$$

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A Measure of Stability Against Singular Perturbations and Robust Properties of Linear Systems

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ABSTRACT

For linear systems the possibility of presenting robust properties through the critical values of the parameter of singular perturbations is shown. Therewith, the nonroughness property is characterized by system stifness. The method of estimating the stiffness is given. All results are given for continuous and discrete systems.

Key words: linear stationary systems, singular perturbations, robustness, stability factors, nonroughness property, system stiffness.

Introduction

In real problems in the mathematical description of the system there is some uncertainty. In the modern control theory parametric methods of describing the uncertainty in the form of interval or affine families of polynomials (regular disturbances of the dynamics), for which, on the basis of the principle of exclusion of zero and the Kharitonov theorem, the robust stability criteria are derived [1], are widely used.

Meanwhile, there are other approaches to describing the uncertainty. Often, the uncertainty in the description of the system can be characterized by singular perturbations [2, 3], i.e., small parameter λ before the derivatives in the Cauchy normal form. Such perturbations in the form of ballast dynamics are a mandatory element of any practical implementation of any control. It has long been known and widely used in real design [4, 5]. Thus, the normal work area of the industrial controller is characterized by the value of the time constant of the ballast link, raising the dimension of a closed system.

Description of the robust properties of the system in the approach, based on singular perturbations, rests on Klimushev–Krasovskiy theorem [6], which discrete analog is used for discrete systems [7]. It is obvious another advantage of the proposed approach: it is known [8] that the result of the Kharitonov theorem type does not hold in the discrete case. Therefore, currently there are no effective means of analysis and synthesis of robust discrete control systems on the basis of the interval method of describing the uncertainty.

The Klimushev–Krasovskiy theorem actually confirms the existence of asymptotically stable family of the systems, parameterized by the values λ from the interval $\{\lambda : 0 < \lambda < \overline{\lambda}\}$. For various $\overline{\lambda}$ let us denote $\Lambda = \{\overline{\lambda} : 0 < \lambda < \overline{\lambda}\}$. The finite value $\lambda_* = \sup \Lambda$ is critical in the sense that for sufficiently small $\varepsilon > 0$ for $\lambda = \lambda_* + \varepsilon$ the system becomes unstable (for unlimited Λ one accepts $\lambda_* = \infty$). Thus, λ_* can be considered, on the one hand, as a measure of stability to singular perturbations, and on the other — as a characteristic of the nonroughness or nonstiffness [2, 3]. It is natural to call value $\vartheta_* = 1/\lambda_*$, inverse to it, the stiffness. It characterizes the structural nonroughness of the system. So, the task is to determine the critical value of the parameter of the singular perturbations λ_* and the stiffness value ϑ_*, corresponding to it.

It is clear that for nonlinear systems, generally, it is practically impossible to determine the exact value λ_*. The existing results can be found in [9]. But for linear stationary systems the problem of determining the critical value λ_* is completely solvable and there are different approaches to its solution. Those ones, which enable to get the exact value, will be called. In [10] the critical value λ_* was determined on the basis of constructing the amplitude phase frequency response function (APFRF) of some matrix function $M(j\omega)$. Later this result was obtained by other investigators on the basis of the Möbius transformation (*LFT*-conversion) [11, 12]. The APFRF method is graphical and not analytical; in addition, the complexity of its implementation depends on the dimension of the "fast" component of the state.

Another approach, to which this work is devoted, is based on the *D*-partitioning method of the characteristic polynomial of a closed system by the parameter of singular perturbations λ. The existing results for continuous systems give the exact value of stiffness in the case when the characteristic equation is given explicitly when the dimension of the "fast" component k is not larger than two [2, 3]. For other cases, they are an asymptotic approximation of the desired values. For discrete systems, a similar result for $k = 1$ was obtained in [13], and for $k = 2$ the stiffness estimate is defined only for the particular case, which will be discussed below. In this paper, the results for determining the exact value of stiffness for continuous and discrete systems with the explicitly given characteristic equation, the

dimension of the "fast" component being $k \leq 3$, are obtained, and also the graphical method for finding the stiffness for arbitrary k is developed. The main advantage of the D-partitioning method therewith consists in the possibility of obtaining an analytical result.

1. The problem formulation of D-partitioning by the parameter of singular perturbations

Let us consider the continuous linear singularly perturbed system

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$A = \begin{pmatrix} A_{00} & A_{01} \\ A_{10}\lambda^{-1} & A_{11}\lambda^{-1} \end{pmatrix}, \quad B = \begin{pmatrix} B_0 \\ B_1\lambda^{-1} \end{pmatrix}, \tag{1}$$

where $x = (x_0^{\mathrm{T}}, x_1^{\mathrm{T}})^{\mathrm{T}}$, $x \in \mathfrak{R}^n$, $x_0 \in \mathfrak{R}^m$, $x_1 \in \mathfrak{R}^k$, $n = m + k$; A is $(n \times n)$-matrix; B is $(n \times r)$-matrix; λ is a positive small parameter (hereinafter T means transposition).

Under the assumption of the controllability of the pair (A, B), the problem of choice of control as a linear form by the state $u(t) = Gx(t)$ is equivalent to definition of the polynomial coefficients in the characteristic equation of system (1)

$$P_{m+k}(s) = s^{m+k} + a_{m+k-1}\vartheta s^{m+k-1} + \ldots + a_{m+1}\vartheta^{k-1}s^{m+1} + \vartheta^k M(s) = 0, \tag{2}$$

where $\vartheta = 1/\lambda$, $M(s) = a_m s^m + a_{m-1}s^{m-1} + \ldots + a_1 s + a_0$.

The Klimushev–Krasovskiy theorem implies [6] that if the "fast" subsystem with the characteristic polynomial $F(s) = s^k + a_{m+k-1}s^{k-1} + \ldots + a_{m+1}s + a_m$ satisfies the Gurvits criterion, the dynamics of the "slow" variables is approximated by the degenerate system, obtained from (1) for $\lambda = 0$ with the "external" characteristic polynomial $S(s) = M(s)/a_m$ [2]. Thus, if the polynomials $F(s)$ and $S(s)$ are stable, then there is such $\lambda_* > 0$, that $\forall \lambda : 0 < \lambda \leq \lambda_*$ the polynomial $P_{m+k}(s)$ is stable.

In the discrete case we consider a linear singularly perturbed system of the form [7]

$$x(k+1) = Ax(k) + Bu(k),$$
$$A = \begin{pmatrix} A_{11} & A_{12} \\ \lambda A_{21} & \lambda A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ \lambda B_2 \end{pmatrix}. \tag{3}$$

Under the assumption of the controllability of the pair (A, B), the problem of choice of control as a linear form by the state $u(k) = Gx(k)$ is equivalent to definition of the polynomial coefficients of the characteristic equation of system (3)

$$P_{m+k}(z) = z^{m+k} + a_{m+k-1}\lambda z^{m+k-1} + \ldots + a_{m+1}\lambda^{k-1}z^{m+1} + \lambda^k M(z) = 0, \tag{4}$$

where $M(z) = a_m z^m + a_{m-1}z^{m-1} + \ldots + a_1 z + a_0$.

In such problem formulation, from [7] it follows that due to the stability of the "external" polynomial $S(z) = z^m$ (the roots of the polynomial lie within a circle of unit radius), the stability of the "fast" polynomial $F(z) = M(z)/a_m$ implies the existence of such $\lambda_* > 0$, that $\forall \lambda : 0 < \lambda \leq \lambda_*$ the polynomial $P_{m+k}(z)$ is stable [7]. The problem again is reduced to finding the critical value of the parameter of singular perturbations λ_*.

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2. Assessment of the second-order stiffness

2.1. The continuous case. When the dimension of the "fast" component of the state is $k = 2$, characteristic equation (2) becomes

$$P_{m+2}(s) = s^{m+2} + a_{m+1}\vartheta s^{m+1} + \vartheta^2 M(s) = 0. \tag{5}$$

In [2, 3], the stiffness ϑ_2 (the lower index shows the order of evaluation) is calculated by the D-partitioning method by the nonlinear parameter ϑ, i.e., first, the equation (5) is solved relative to ϑ, and then to the resulting expression the D-partitioning method was applied. As a result for ϑ_2 in [2, 3], one has:

$$\vartheta_2 = \frac{1}{\lambda_*} = \max_{\overline{\omega}:N_R(\overline{\omega})=0} \frac{-\overline{\omega} V_m(\overline{\omega})}{a_{m+1} R_m(\overline{\omega})}, \quad m = 2\nu,$$

$$\vartheta_2 = \frac{1}{\lambda_*} = \max_{\overline{\omega}:N_V(\overline{\omega})=0} \frac{\overline{\omega} R_m(\overline{\omega})}{a_{m+1} V_m(\overline{\omega})}, \quad m = 2\nu + 1, \tag{6}$$

where

$$N_R = (-1)^\nu R_m(\overline{\omega}) a_{m+1}^2 \overline{\omega}^m - V_m^2(\overline{\omega}),$$

$$N_V = (-1)^\nu V_m(\overline{\omega}) a_{m+1}^2 \overline{\omega}^m - R_m^2(\overline{\omega}),$$

$R_m(\omega)$ and $V_m(\omega)$ are real and imaginary parts of the polynomial $M(j\omega)$, respectively.

First, note that this estimate ϑ_2 can be simplified. Let us express the last relation $V_m(\omega)/R_m(\omega)$:

$$V_m(\omega)/R_m(\omega) = (-1)^\nu a_{m+1}^2 / V_m(\omega), \quad m = 2\nu,$$

$$R_m(\omega)/V_m(\omega) = (-1)^\nu a_{m+1}^2 / R_m(\omega), \quad m = 2\nu + 1.$$

Substituting the obtained relations in (8), one obtains more simple relations for estimating ϑ_2:

$$\vartheta_2 = \max_{\overline{\omega}:N_R(\overline{\omega})=0} \frac{(-1)^{\nu+1} a_{m+1} \overline{\omega}^{m+1}}{V_m(\overline{\omega})}, \quad m = 2\nu,$$

$$\vartheta_2 = \max_{\overline{\omega}:N_V(\overline{\omega})=0} \frac{(-1)^\nu a_{m+1} \overline{\omega}^{m+1}}{R_m(\overline{\omega})}, \quad m = 2\nu + 1.$$

Let us consider now the problem of finding ϑ_2 by the D-partitioning method in the plane of two real parameters $\{\xi_1, \xi_2\}$: $\xi_1 = \lambda$, $\xi_2 = \lambda^2$. To do this, let us reduce characteristic equation (7) to the form $\widetilde{P}_{m+2}(s) = \xi_2 P(s) + \xi_1 Q(s) + M(s) = 0$, where $P(s) = s^{m+2}$, $Q(s) = a_{m+1}s^{m+1}$.

Furthermore, in accordance with the D-partitioning technique [14], introducing the substitution $s = j\omega$ in the transformed characteristic equation, one obtains the system, containing two equations: $\operatorname{Re} P_{m+2}(j\omega, \xi_1, \xi_2) = 0$ and $\operatorname{Im} P_{m+2}(j\omega, \xi_1, \xi_2) = 0$. From their solution one obtains the expressions for the parameters $\xi_1 = \xi_1(\omega)$, $\xi_2 = \xi_2(\omega)$:

$$\xi_1(\omega) = \frac{(-1)^{\nu+1} V_m(\omega)}{a_{m+1}\omega^{m+1}}, \quad \xi_2(\omega) = \frac{(-1)^\nu R_m(\omega)}{\omega^{m+2}}, \quad m = 2\nu,$$

$$\xi_1(\omega) = \frac{(-1)^\nu R_m(\omega)}{a_{m+1}\omega^{m+1}}, \quad \xi_2(\omega) = \frac{(-1)^\nu V_m(\omega)}{\omega^{m+2}}, \quad m = 2\nu + 1, \tag{7}$$

where $R_m(\omega) = \mathrm{Re}\, M(j\omega)$, $V_m(\omega) = \mathrm{Im}\, M(j\omega)$. Relationships (7) have a singularity at the point $\omega = 0$, the singular line

$$\xi_2 = 0. \tag{8}$$

corresponds to this frequency value. Expressions (7), (8) define in the plane $\{\xi_1, \xi_2\}$ the stability region boundary D. In Figure 1, this region is highlighted by thick lines. The critical parameter value of singular perturbations is defined as follows: $\lambda_* = \min_{\overline{\omega} \in \Omega} \xi_1(\overline{\omega})$, where Ω is a set of the frequencies $\overline{\omega}_i$, which satisfy the condition

$$\xi_1^2(\overline{\omega}_i) = \xi_2(\overline{\omega}_i). \tag{9}$$

Condition (9) follows from the adopted notations, it corresponds to the points of intersection of the curve of the D-partitioning with the parabola $\xi_1^2(\overline{\omega})$ in the parameter plane $\{\xi_1, \xi_2\}$ (see Figure 1).
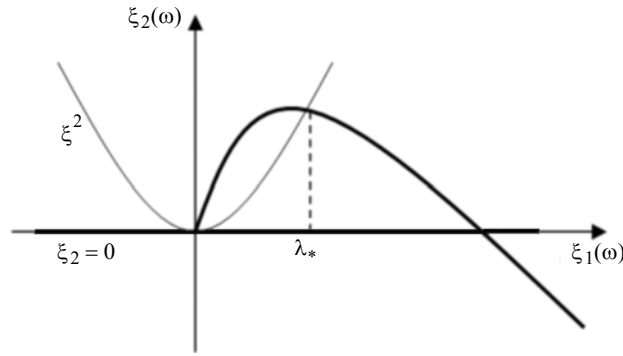

Figure 1

Substituting (7) in condition (9), we write the final expression, defining the set of frequencies Ω:

$$\Omega = \{\overline{\omega} : N_R(\overline{\omega}) = 0\}, \quad m = 2\nu, \quad \Omega = \{\overline{\omega} : N_V(\overline{\omega}) = 0\}, \quad m = 2\nu + 1.$$

Finally, for assessing the stiffness of the second order we obtain

$$\vartheta_2 = 1/\lambda_* \frac{(-1)^{\nu+1} a_{m+1} \overline{\omega}^{m+1}}{V_m(\overline{\omega})}, \quad m = 2\nu,$$

$$\vartheta_2 = 1/\lambda_* = \max_{\overline{\omega} : N_V(\overline{\omega}) = 0} \frac{(-1)^{\nu} a_{m+1} \overline{\omega}^{m+1}}{R_m(\overline{\omega})}, \quad m = 2\nu + 1, \tag{10}$$

which coincides with the results of [2, 3].

2.2. The discrete case. Let us write the characteristic equation (4) for $k = 2$

$$P_{m+2}(z) = z^{m+2} + a_{m+1}\lambda z^{m+1} + \lambda^2 M(z) = 0.$$

By analogy with the continuous case, we will reduce this characteristic equation to the form suitable for application of the method of D-partitioning by two parameters:

$$\widetilde{P}_{m+2}(z) = \xi_2 P(z) + \xi_1 Q(z) + M(z) = 0. \tag{11}$$

Here $P(z) = z^{m+2}$, $Q(z) = a_{m+1} z^{m+1}$, $\xi_1 = \vartheta = 1/\lambda$, $\xi_2 = \vartheta^2 = 1/\lambda^2$. Substituting $z = e^{j\omega}$ in (11), we find expressions for the parameters in the form $\xi_1 = \xi_1(\omega)$, $\xi_2 = \xi_2(\omega)$:

$$\xi_1(\omega) = -\frac{a_m \sin 2\omega + \ldots + a_0 \sin(m+2)\omega}{a_{m+1} \sin \omega},$$

$$\xi_2(\omega) = \frac{a_m \sin \omega + \ldots + a_0 \sin(m+1)\omega}{\sin \omega}.$$

(12)

At frequencies $\omega = 0$ and $\omega = \pi$ relationships (12) have singularities, so for these frequencies singular lines are constructed:

$$\xi_2 R_p(\omega) + \xi_1 R_q(\omega) + R_m(\omega) = 0, \ \omega = 0, \pi,$$

$$R_m(\omega) = \operatorname{Re} M(e^{j\omega}), \ R_p(\omega) = \operatorname{Re} P(e^{j\omega}), \ R_q(\omega) = \operatorname{Re} Q(e^{j\omega}).$$

(13)

Expressions (12), (13) define the boundaries of the stability domain of the characteristic polynomial $P_{m+2}(z)$ in the plane $\{\xi_1, \xi_2\}$. The critical value of stiffness is equal to $\vartheta_* = \max_{\overline{\omega} \in \Omega} \xi_1(\overline{\omega})$, where Ω is a set of the frequencies $\overline{\omega}_i$, which satisfy condition (9).

Condition (9), as well as in the continuous case, can be interpreted as a set of points of intersection of the D-partitioning curve with the parabola $\xi_1^2(\overline{\omega})$ (Figure 2). Substituting (12) into (9), one will have for the set Ω

$$\Omega = \{\overline{\omega} : a_{m+1}^2 \sin \overline{\omega} (a_m \sin \overline{\omega} + \ldots + a_0 \sin(m+1)\overline{\omega}) - (a_m \sin 2\overline{\omega} + \ldots + a_0 \sin(m+2)\overline{\omega})^2 = 0\}. \quad (14)$$

Note, that if the parabola $\xi_1^2(\overline{\omega})$ intersects the D-partitioning curve in the points of intersection with the singular lines (as shown in Figure 3), i.e., $\overline{\omega}_1 = 0$ or $\overline{\omega}_2 = \pi$ are roots of (14), then the critical value of stiffness can not be determined from (12), because at these frequency values the expression for $\xi_1(\overline{\omega}_i)$ has singularities. In such cases, the critical value of stiffness will be determined as $\vartheta_* = \max_{\overline{\omega} \in \{0, \pi\}} \overline{\xi}_1(\overline{\omega})$, where $\overline{\xi}_1(\overline{\omega})$ is a solution of the following equations:

$$\overline{\xi}_1^2 R_p(\overline{\omega}) + \overline{\xi}_1 R_q(\overline{\omega}) + R_m(\overline{\omega}) = 0, \ \overline{\omega} = 0, \pi. \quad (15)$$

Taking into account (12), (14), (15), to assess the stiffness of the second order we will write finally:

$$\vartheta_2 = \max_{\overline{\omega} \in \Omega} \frac{-a_m \sin 2\overline{\omega} - \ldots - a_0 \sin(m+2)\overline{\omega}}{a_{m+1} \sin \overline{\omega}}, \ \overline{\omega} \neq 0, \pi,$$

$$\vartheta_2 = \max_{\overline{\omega} \in \Omega} \frac{-R_q(\overline{\omega}) \pm \sqrt{R_q(\overline{\omega}) - 4R_p(\overline{\omega}) R_m(\overline{\omega})}}{2R_p(\overline{\omega})}, \ \overline{\omega} = 0, \pi,$$

(16)

set Ω is defined by (14).

Relationships (16) uniquely determine the assessment of the second-order stiffness for discrete systems, in contrast to [13], where the cases $\overline{\omega} = 0, \pi$ have nor been considered. Note that when $\overline{\omega} = 0$ and (or) $\overline{\omega} = \pi$, for estimating (16) the analytical representation for arbitrary dimension of the "slow" component can be obtained.
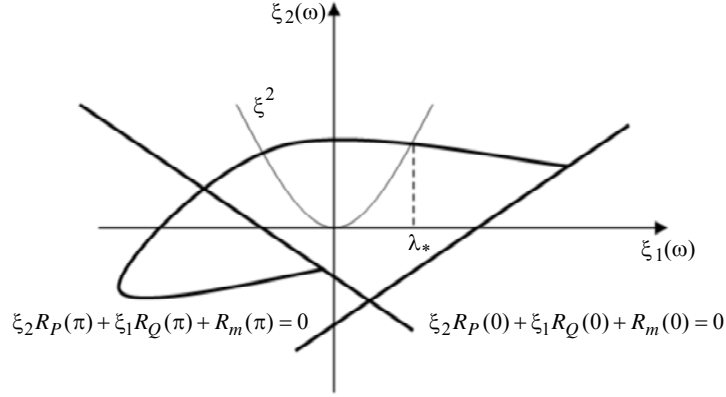
$$\xi_2 R_P(\pi) + \xi_1 R_Q(\pi) + R_m(\pi) = 0 \qquad \xi_2 R_P(0) + \xi_1 R_Q(0) + R_m(0) = 0$$

Figure 2

$$\xi_2 R_P(\pi) + \xi_1 R_Q(\pi) + R_m(\pi) = 0 \qquad \xi_2 R_P(0) + \xi_1 R_Q(0) + R_m(0) = 0$$
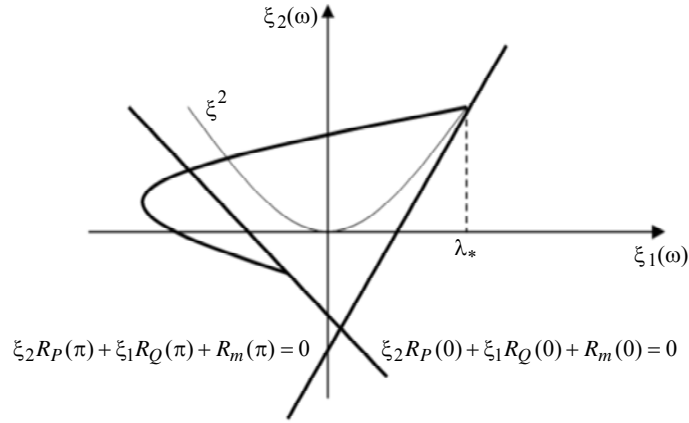
Figure 3

3. Assessment of the third order stiffness

3.1. The continuous case. Let us consider the characteristic equation (2) of the singularly perturbed system (1) for $k = 3$:

$$P_{m+3}(s) = s^{m+3} + a_{m+2}\vartheta s^{m+2} + a_{m+1}\vartheta^2 s^{m+1} + \vartheta^3 M(s) = 0. \tag{17}$$

Let us reduce this equation to the form

$$\widetilde{P}_{m+3}(s) = \xi_2\xi_1 L(s) + \xi_2 P(s) + \xi_1 Q(s) + M(s) = 0,$$

where $L(s) = s^{m+3}$, $P(s) = a_{m+2}s^{m+2}$, and the rest of the notations correspond to the ones, taken earlier. Using the *D*-partitioning technique, one obtains the following expressions for the parameters ξ_1 and ξ_2:

$$\xi_1(\omega) = \frac{V_m(\omega)a_{m+2}}{R_m(\omega)\omega + (-1)^{\nu+1}a_{m+2}a_{m+1}\omega^{m+1}}, \quad \xi_2(\omega) = \frac{(-1)^\nu R_m(\omega)}{a_{m+2}\omega^{m+2}} \text{ for } m = 2\nu;$$

$$\xi_1(\omega) = \frac{-R_m(\omega)a_{m+2}}{V_m(\omega)\omega + (-1)^{\nu+1}a_{m+2}a_{m+1}\omega^{m+1}}, \quad \xi_2(\omega) = \frac{(-1)^\nu V_m(\omega)}{a_{m+2}\omega^{m+2}} \text{ for } m = 2\nu+1. \tag{18}$$

Also, there is a singular line $\xi_2 = 0$, which together with (18), defines the boundaries of the stability domain of characteristic polynomial (17) in the plane $\{\xi_1, \xi_2\}$. As before, the critical parameter value of singular perturbations is defined as $\lambda_* = \min\limits_{\overline{\omega} \in \Omega} \xi_1(\overline{\omega})$, where Ω is a set of the frequencies $\overline{\omega}_i$, which satisfy condition (9). In the notations $R_m(\omega)$ and $V_m(\omega)$, this condition is as follows:

$$N_R(\overline{\omega}) = R_m^2(\overline{\omega})\, a_{m+2}^3 \overline{\omega}^{m+2} + 2V_m^2(\overline{\omega})\, a_{m+1} a_{m+2} \overline{\omega}^{m+2} +$$

$$+ (-1)^{\nu+1}(V_m^3(\overline{\omega})\, \overline{\omega}^2 + V_m(\overline{\omega})\, a_{m+1}^2 a_{m+2}^2 \overline{\omega}^{2m+2}), \ \ m = 2\nu, \tag{19}$$

$$N_V(\overline{\omega}) = V_m^2(\overline{\omega})\, a_{m+2}^3 \overline{\omega}^{m+2} + 2R_m^2(\overline{\omega})\, a_{m+1} a_{m+2} \overline{\omega}^{m+2} +$$

$$+ (-1)^{\nu+1}(R_m^3(\overline{\omega})\, \overline{\omega}^2 + R_m(\overline{\omega})\, a_{m+1}^2 a_{m+2}^2 \overline{\omega}^{2m+2}), \ \ m = 2\nu + 1. \tag{20}$$

Taking into account that $\lambda_* = \min\limits_{\overline{\omega} \in \Omega} \xi_1(\overline{\omega})$, for estimating the third order stiffness, we obtain

$$\vartheta_3 = \max\limits_{\overline{\omega}:N_R(\overline{\omega})=0} \frac{R_m(\overline{\omega})\, \overline{\omega} + (-1)^{\nu+1} a_{m+2} a_{m+1} \overline{\omega}^{m+1}}{V_m(\omega)\, a_{m+2}}, \ \ m = 2\nu,$$

$$\vartheta_3 = \max\limits_{\overline{\omega}:N_V(\overline{\omega})=0} -\frac{V_m(\overline{\omega})\, \overline{\omega} + (-1)^{\nu+1} a_{m+2} a_{m+1} \overline{\omega}^{m+1}}{R_m(\overline{\omega})\, a_{m+2}}, \ \ m = 2\nu + 1,$$

where $N_R(\overline{\omega})$ and $N_V(\overline{\omega})$ are determined by formulas (19), (20).

3.2. The discrete case. The case $k = 3$ reduces characteristic equation (4) to the form

$$P_{m+3}(z) = z^{m+3} + a_{m+2}\lambda z^{m+2} + a_{m+1}\lambda^2 z^{m+1} + \lambda^3 M(z) = 0. \tag{21}$$

By analogy with the continuous case, let us represent (21) in the form suitable for application of the *D*-partitioning method

$$\widetilde{P}_{m+3}(z) = \xi_1\xi_2 L(z) + \xi_2 P(z) + \xi_1 Q(z) + M(z) = 0.$$

Here $L(z) = z^{m+3}$, $P(z) = a_{m+2} z^{m+2}$, and the remaining notations, correspond to the ones, taken earlier. The significant difference of the discrete case from the continuous one lies in the fact that the *D*-partitioning equations, obtained by substituting $z = e^{j\omega}$, depend both on ξ_1 and ξ_2, and this dependence is nonlinear. Let us write them:

$$\operatorname{Re}\widetilde{P}_{m+3}(e^{j\omega}) = \xi_1\xi_2 R_l(\omega) + \xi_2 R_p(\omega) + \xi_1 R_q(\omega) + R_m(\omega) = 0,$$

$$\operatorname{Im}\widetilde{P}_{m+3}(e^{j\omega}) = \xi_1\xi_2 V_l(\omega) + \xi_2 V_p(\omega) + \xi_1 V_q(\omega) + V_m(\omega) = 0.$$

Solving the last equation for ξ_1 and ξ_2, one gets

$$\xi_1(\omega) = \frac{-b(\omega) \pm \sqrt{b(\omega)^2 - 4a(\omega)c(\omega)}}{2a(\omega)},$$

$$\xi_2(\omega) = \frac{\xi_1(\omega)R_q(\omega) - R_m(\omega)}{R_p(\omega) + \xi_1(\omega)R_l(\omega)}, \tag{22}$$

where

$$a(\omega) = -a_{m+1} \sin \omega, \quad b(\omega) = -a_{m+1} a_{m+2} \sin \omega,$$

$$c(\omega) = -a_{m+2}(a_0 \sin(m+2)\omega + \ldots + a_m \sin 2\omega).$$

For $\omega = 0, \pi$ from (21) one obtains the equations of the singular curves:

$$\xi_1 \xi_2 R_l(0) + \xi_2 R_p(0) + \xi_1 R_q(0) + R_m(0) = 0,$$

$$\xi_1 \xi_2 R_l(\pi) + \xi_2 R_p(\pi) + \xi_1 R_q(\pi) + R_m(\pi) = 0. \tag{23}$$

Together with (22) they determine the stability domain of polynomial (21) in space $\{\xi_1, \xi_2\}$. The desired for us critical parameter value of singular perturbations is determined by the point of intersection of the parabola $\xi_1^2(\overline{\omega})$ with the boundary of the stability domain in the space $\{\xi_1, \xi_2\}$. Then the estimate of the third order stiffness will be determined by the expression

$$\vartheta_3 = \max_{\overline{\omega} \in \Omega} \frac{-b(\overline{\omega}) \pm \sqrt{b(\overline{\omega})^2 - 4a(\overline{\omega})c(\overline{\omega})}}{2a(\overline{\omega})}, \tag{24}$$

where the set Ω is found from condition (9).

4. Estimates of the higher orders stiffness

It is clear that with increasing the order of the estimate, the process of its finding is complicated, and obtaining an analytical representation for the stiffness becomes more and more difficult, but this does not exclude the possibility of its numerical determination. Let us show a graphical method for determining the stiffness for continuous systems with arbitrary "fast" and "slow" components of a state.

Let us consider characteristic equation (2), in which we will denote $\xi_1 = \lambda, \ \xi_2 = \lambda^2$. Introducing the substitution $s = j\omega$, depending on k and m, we will get one of the following systems:

- for $k = 2\mu + 1, \ m = 2\nu$

$$\mathrm{Re}\,\widetilde{P}_{m+k}(j\omega) = (-1)^{\nu+\mu} \xi_2^{\mu} \omega^{m+k-1} + \ldots + (-1)^{\nu+1} a_{m+2} \xi_2 \omega^{m+2} + R_m(\omega),$$

$$\mathrm{Im}\,\widetilde{P}_{m+k}(j\omega) = (-1)^{\nu+\mu} \xi_1 \xi_2^{\mu} \omega^{m+k} + \ldots + (-1)^{\nu} a_{m+1} \xi_1 \omega^{m+1} + V_m(\omega);$$

- for $k = 2\mu + 1, \ m = 2\nu + 1$

$$\mathrm{Re}\,\widetilde{P}_{m+k}(j\omega) = (-1)^{\nu+\mu+1} \xi_1 \xi_2^{\mu} \omega^{m+k} + \ldots + (-1)^{\nu+1} a_{m+1} \xi_1 \omega^{m+1} + R_m(\omega),$$

$$\mathrm{Im}\,\widetilde{P}_{m+k}(j\omega) = (-1)^{\nu+\mu} \xi_2^{\mu} \omega^{m+k-1} + \ldots + (-1)^{\nu+1} a_{m+2} \xi_2 \omega^{m+2} + V_m(\omega);$$

- for $k = 2\mu, \ m = 2\nu$ \hfill (25)

$$\mathrm{Re}\,\widetilde{P}_{m+k}(j\omega) = (-1)^{\nu+\mu} \xi_2^{\mu} \omega^{m+k} + \ldots + (-1)^{\nu+1} a_{m+2} \xi_2 \omega^{m+2} + R_m(\omega),$$

$$\mathrm{Im}\,\widetilde{P}_{m+k}(j\omega) = (-1)^{\nu+\mu-1} \xi_1 \xi_2^{\mu} \omega^{m+k-1} + \ldots + (-1)^{\nu} a_{m+1} \xi_1 \omega^{m+1} + V_m(\omega);$$

- for $k = 2\mu$, $m = 2\nu + 1$

$$\mathrm{Re}\,\widetilde{P}_{m+k}(j\omega) = (-1)^{\nu+\mu}\xi_1\xi_2^{\mu}\omega^{m+k} + \ldots + (-1)^{\nu+1}a_{m+1}\xi_1\omega^{m+1} + R_m(\omega),$$

$$\mathrm{Im}\,\widetilde{P}_{m+k}(j\omega) = (-1)^{\nu+\mu-1}\xi_2^{\mu}\omega^{m+k-1} + \ldots + (-1)^{\nu+1}a_{m+2}\xi_2\omega^{m+2} + V_m(\omega).$$

The distinctive feature of equations (25) is that, regardless of k and m, one of the equations depends on ξ_1 and ξ_2, moreover on ξ_1 linearly, and another — only on ξ_2 and this dependence is nonlinear. Based on this, let us express the parameter ξ_1 in terms of ξ_2, and the parameter ξ_2 we find through solving the power equation. Thus, obtaining the expressions for the parameters $\xi_1(\omega)$ and $\xi_2(\omega)$, one can construct the D-partitioning curve. Then the parameter critical value of the singular perturbations will be determined by intersection of the D-partitioning curve with the parabola $\xi_1^2(\overline{\omega})$. Below the examples, showing the efficiency of this approach, are given.

Note. Further efforts to develop the D-partitioning method for determining the stiffness of stationary systems can be aimed at simplification of the obtained algorithms. In this direction, probably, new developments can be obtained by using the recent results on the D-partitioning method for polynomial families of a special form [15, 16].

5. Examples of definition of stiffness

5.1. The continuous case. Suppose, that the system has the two-dimensional "slow" and a two-dimensional "fast" components of the state, with the following characteristic polynomial of the system:

$$P_4(s) = s^4 + 4.25\vartheta s^3 + \vartheta^2(3.45s^2 + 3.87s + 1.4).$$

The estimation of stiffness is $\vartheta_2 = \max\limits_{\overline{\omega}\in\Omega} 1.099\,\overline{\omega}^2$. In this case $\Omega = \{-0.8, 0.8\}$, then $\vartheta_2 = 0.704$. The roots r_i of the characteristic equation for $\vartheta = \vartheta_2$ are: $r_{1,2} = \pm 0.80\,j$, $r_3 = -0.425$, $r_4 = -2.614$. The system is on the border of stability.

Let us consider a system with $k = 2$, $m = 4$. The characteristic equation has the form

$$P_6(s) = s^6 + 9.1\,\vartheta s^5 + \vartheta^2(30.4s^4 + 47.0\,s^3 + 38.4s^2 + 16.0s + 2.4) = 0.$$

The stiffness $\vartheta = \max\limits_{\overline{\omega}\in\Omega}\left(\dfrac{\overline{\omega}^5}{-5.131\overline{\omega}^3 + 1.75\overline{\omega}}\right)$, $\Omega = \{1.3, -1.3\}$, one obtains $\vartheta_2 = 0.429$. The roots r_i: $r_{1,2} = -0.44 \pm 0.33i$, $r_3 = -1.33i$, $r_4 = 1.33i$, $r_5 = -0.30$, $r_6 = -2.73$. The system is on the border of stability.

Let us consider a system for $k = 3$, $m = 2$. The characteristic equation has the form

$$P_5(s) = s^5 + 8.1\,\vartheta s^4 + 6.3\,\vartheta^2 s^3 + \vartheta^3(38.09s^2 + 7.7s + 19.5) = 0.$$

The stiffness $\vartheta = \max\limits_{\overline{\omega}\in\Omega}\left(\dfrac{\overline{\omega}(2.38 - 4.65\overline{\omega}^2) + 6.3\overline{\omega}^3}{7.7\overline{\omega}}\right)$, $\Omega = \{1.5, -1.5\}$, one gets $\vartheta_2 = 0.789$. Roots r_i: $r_{1,2} = \pm 1.5\,j$, $r_{3,4} = -0.08 \pm 0.82\,j$, $r_5 = -6.30$. The system is on the border of stability.

Let us analyze the systems of a high order. Let $k = 8$, $m = 2$. Let us write the characteristic polynomial as:

$$P_{10}(s) = s^{10} + 12.82\vartheta\, s^9 + 75.06\,\vartheta^2 s^8 + 265.92\vartheta^3 s^7 + 604.49\,\vartheta^4 s^6 +$$
$$+ 859.59^5 s^5 + 726.89^6 s^4 + 344.59^7 s^3 + \vartheta^8(85.04\, s^2 + 10.03\, s + 0.43).$$

It is clear, that it is difficult to determine numerically the set Ω of frequencies, for which the D-partitioning hodograph intersects with the parabola $\xi_1^2(\overline{\omega})$. But it is not necessary, the parameter critical value of the singular perturbations can be determined graphically (Figure 4: solid line — D-partitioning hodograph; dotted line —parabola $\xi_1^2(\overline{\omega})$).
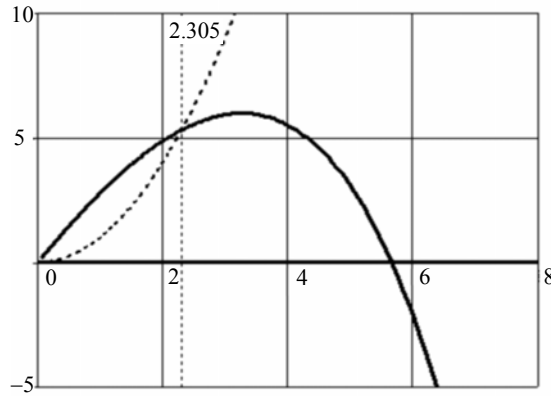
Figure 4

The graphic way showed that $\lambda_* = 2.305$, therefore, the stiffness $\vartheta_8 = 0.433$. The roots r_i: $r_{1,2} = -0.00033 \pm 0{,}12j$, $r_3 = -0.0622$, $r_{4,5} = -0.38 \pm 0{,}27j$, $r_{6,7} = -0.59 \pm 1{,}1j$, $r_{8,9} = -0.97 \pm 0.099j$, $r_{10} = -1.58$. The roots $r_{1,2}$ practically lie on the imaginary axis, then $\lambda_* = 2.305$ can be considered as a measure of the "roughness" of the system.

5.2. The discrete case. Let for the system $k = 2, m = 2$ one has the characteristic polynomial

$$P_4(z) = z^4 + \lambda\, 0.548 z^3 + \lambda^2 (0.38 z^2 - 0.197z - 0.023).$$

In this case $\Omega = \{-2.1, 2.1\}$, then $\vartheta_2 = \max\limits_{\overline{\omega} \in \Omega}\left(\dfrac{-a_2 \sin 2\overline{\omega} - a_1 \sin 3\overline{\omega} - a_0 \sin 4\overline{\omega}}{a_3 \sin \overline{\omega}}\right)$. One gets $\vartheta_2 = 0.763$. The roots r_i of the polynomial $P_4(z)$ for $\lambda = 1/\vartheta_3 = 1.31$ are such that $|r_{1,2}| = 1$, $|r_i| < 1$, $i = 3$, 4. The system is on the border of stability.

Let us consider the system for $k = 2$, $m = 8$, its characteristic equation has the form

$$P_{10}(z) = z^{10} - 1.85\lambda z^9 + \lambda^2(0.76 z^8 + 0.2 z^7 - 0.66 z^6 +$$
$$+ 1.034 z^5 + 1.034 z^4 - 0.48 z^3 + 0.056 z^2 - 0.0025 z - 0.001) = 0.$$

In this case, $\Omega = \{0\}$, then $\vartheta_2 = \dfrac{-R_q(0) \pm \sqrt{R_q(0) - 4R_p(0)R_m(0)}}{2R_p(0)} = 0.998$. The roots r_i of the polynomial $P_{10}(z)$ for $\lambda = 1/\vartheta_3 = 1.002$ are such that $|r_1| = 0.999$, $|r_i| < 1$, $i = \overline{2, 10}$. Obviously, for such stiffness value the system is on the stability boundary.

Let us consider the system for $k = 3$, $m = 2$ and the characteristic polynomial

$$P_5(z) = z^5 - 0.483\lambda z^4 - 0.058\lambda^2 z^3 + \lambda^3(0.056z^2 - 0.0025z - 0.001) = 0.$$

The stiffness $\vartheta_2 = 0.789$. The roots r_i of the polynomial $P_5(z)$ for $\lambda = 1/\vartheta_3 = 2.381$ are such that $|r_{1,2}| = 0.997$, $|r_3| = 0.112$, $|r_4| = 0.74$, $|r_5| = 0.17$. The system is on the border of stability.

Conclusion

For linear stationary systems on the basis of the method of singular perturbations the criteria of the robustness and stability factors are constructed. The nonroughness property is determined by the system stiffness. The results both for continuous and discrete linear systems are obtained.

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On Construction of Finite-Dimensional Mathematical Model of Convection–Diffusion Process with Usage of the Petrov–Galerkin Method

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ABSTRACT

We proposed technique of construction of piecewise-polynomial weighting functions for the Petrov–Galerkin method in two-dimensional domain. The form of these functions is defined by finite number of variable parameters, connected with grid edges. We consider several variants of selection of these parameters depending on the modulus and direction of the vector of advection rate. The method of construction of finite-dimensional model of nonstationary process of convection-diffusion for arbitrary domain in the form of system of ordinary differential equations was proposed. Accuracy of the obtained model is ascertained by comparison of numerical and analytical solutions of a testing problem.

Key words: construction of piecewise-polynomial weighting functions, Petrov–Galerkin method, two-dimensional domain, finite number of variable parameters, construction of finite-dimensional model, nonstationary process, convection-diffusion, arbitrary domain, ordinary differential equations, accuracy, comparison of numerical and analytical solutions.

Introduction

Recently methods of cybernetics find growing application in physics. This is connected both with necessity of solving problems of control of complex physical systems [1–4] and treatment and processing of measurement data, namely, refinement of mathematical models, restoration of global form of physical fields by local measurements, solution of prediction problems. The necessary component part of these problems is construction or selection of mathematical model, which describes sufficiently accurately behavior of the investigated system.

For physical systems, described by equations in partial derivatives, one uses as models integral representations of their solutions [5, 6], which are obtained by means of the Green functions and fundamental solutions [7, 8], as well as finite-dimensional approximations of initial equations in the form of ordinary differential equations [9–11]. Every of these approaches has its own advantages and disadvantages, and efficiency of their application is defined finally by definite problem. In particular, on investigation of physical processes in domains of complex shape the usage of the Green functions is difficult, since construction of the Green function in this case represents independent rather complicated problem [12]. Usage of fundamental solutions removes this restriction, however, it is possible to find these solutions easily only for equations with constant coefficients [7, 8, 13]. The approach, which uses finite-dimensional approximations in the form of ordinary differential equations (ODE), is the most universal. Moreover, in this case for solving problems of control and estimation we succeeded in using standard procedures of optimal control and observation [14, 15].

Finite-dimensional models of systems with distributed parameters can be obtained by certain method of numerical solution of equations in partial derivatives [16–27]. At the same time pure mechanical usage of threes methods for obtaining mathematical model in the form of ODE is impossible, since they are developed namely for numerical solution of and have mostly algorithmic character.

One of the most popular methods of numerical solution of differential and integral equations now is the Galerkin method [20–24]. It was suggested at the beginning of XX century by professor of the Marine Academy I.G. Bubnov for solving problems of theory of elasticity. Then, method was enhanced by professor of the Petersburg Polytechnic Institute B.G. Galerkin, and it was generalized for solving arbitrary problems of mathematical physics. New and numerous areas of application of the Galerkin method appeared after suggestion [20] to use as basis function the functions of simple form with finite support [7]. Here the Galerkin method got the name of the finite element method (FEM). It is used successfully for solving problems of elasticity, diffusion and heat conductivity, as well as in hydrodynamics for computation of potential flows [20, 22, 24]. Wide usage of FEM is connected with automation of the process of construction of conservative difference schemes [20, 25] on realization of calculations for domains of complex shape, and with potential of computational process effective paralleling.

On solving practically significant problems of convection-diffusion with prevalent convection numerical solutions, obtained by the Galerkin method are, as a rule, instable or oscillatory for stable analytical solution [20, 21, 26]. They belong to singular problems of mathematical physics problems (small coefficient at the higher derivative) and their solution, including numerical one, represents considerable complexities [27]. There are many modifications of the Galerkin method now [20, 27], which make it possible to overcome this insufficiency. One of the most effective is the Galerkin–Petrov method, which is distinguished by special selection of weighting functions, not coinciding with basis functions [20, 27–35]. For a one-dimensional stationary problem of convection diffusion were found weighting functions such, that their numerical and analytical solutions coincide at mesh points [28].

In the present article we suggest the method for construction of weighting functions of the Galerkin–Petrov method [20, 21] for nonstationary problems of convection and diffusion. By means of these

functions we obtained finite-dimensional model of the process of convection–diffusion, which is applicable for arbitrary simply connected two-dimensional domain. Accuracy of the model was shown on comparison of numerical solution with analytical one for certain problem of heat conductivity with convective addend, which is characterized by great value of the Peclet number.

1. Problem statement

Let us consider peculiarities of numerical solution of problems of convection-diffusion by the example of one-dimensional propagation of heat in medium, which moves relative to immovable reference frame with constant velocity v. This process is described by the following equation

$$\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial x} = \kappa\frac{\partial^2 T}{\partial x^2}, \ \ T = T(t, x), \ \ x \in [0, L], \ \ t \in [0, t_1], \tag{1}$$

where $T = T(t, x)$ is the temperature of medium at the point with coordinate x at time instant t, κ is the coefficient of temperature conductivity.

The Galerkin method makes it possible to obtain approximate weak solution [9, 36] of this equation. Here solution is searched as

$$\hat{T}(t, x) = \sum_{i=0}^{n} a_i(t)N_i(x). \tag{2}$$

Here $N_i(x)$ are known the so-called basis functions. Relations for determination of unknown coefficients of decomposition $a_i(t)$ in (2) are obtained in the following way. Let us substitute the expression for $\hat{T}(t, x)$ into the equation (1), multiply the obtained equality by the function $N_j(x)$ ($j = \overline{1, n-1}$), which is accepted in this case to be called as weighting function, and integrate the obtained equality over spatial variable x on the interval $[0, L]$. As the result we obtain the system of ODE for determination of the coefficients $a_i(t)$. Initial conditions for the equation (1) define initial conditions for this system, while boundary conditions enter its right-hand part [9, 20–24]. For numerical solution it is necessary to integrate the obtained system of ODE.

We can take the simplest piecewise-linear functions as the functions $N_i(x)$. They are determined in the following way. Let us select on the interval $[0, L]$ points x_i, $i = \overline{0, n}$, such, that $x_i < x_{i+1}$, moreover, $x_0 = 0$ and $x_n = L$. One supposes that these points set grid on the interval $[0, L]$, therefore they are called nodes. We take continuous positive function $N_i(x)$, different from zero only on the interval $[x_{i-1}, x_{i+1}]$, linear on the intervals $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ and equal to 1 at the point x_i. On usage of such functions, which differ from zero only on small element $[x_{i-1}, x_{i+1}]$ of the domain of solution, the Galerkin method is usually called the finite element method.

Solving the equation (1) by means of the described procedure under the absence of convective addend, $v = 0$, does not cause complexities.

Let us consider peculiarities of numerical solution of this equation under the presence of convective addend. Here we suppose that diffusion addend is absent, $\kappa = 0$. In this case the equation (1) takes the form

$$\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial x} = 0. \tag{3}$$

The general solution of this equation is

$$T(t, x) = g(x - vt), \tag{4}$$

where $g(\cdot)$ is arbitrary continuously-differentiable function [37]. Graph of solution for time instant $t + \Delta t$, as it follows from (4), is obtained by shifting the graph for time instant t by the value $v\Delta t$ along the axis x. If we continuously observe this process in time, then we obtain that graph of the function $g(\cdot)$ moves with the velocity v along the axis x. Graphs of solution of the equation (3), which correspond to different time instants, are shown in Figure 1, arrow indicates direction of velocity. The solution (4) is called the running wave [17, 37], and the equation (3) is called the transfer equation.
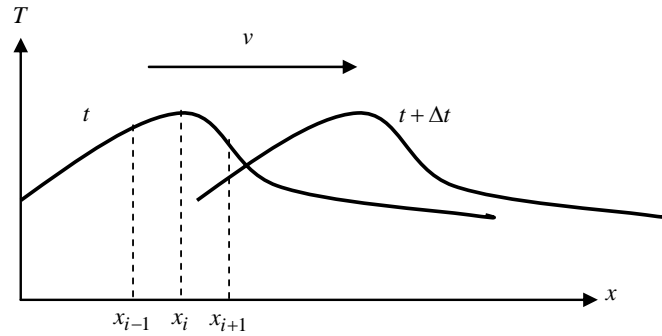
Figure 1

Usage of the described above procedure and functions $N_i(x)$ for numerical solution of the equation (3) under the condition that nodes on the interval $[0, L]$ are selected as equidistant $x_{i+1} - x_i = h$, results in the following equations for coefficients

$$\frac{1}{6}(\dot{a}_{j-1} + 4\dot{a}_j + \dot{a}_{j+1}) = -v\frac{a_{j+1} - a_{j-1}}{2h}, \quad j = \overline{1, n-1}. \tag{5}$$

The equations for boundary nodes are written taking into account boundary conditions and here we do not adduce them. Solutions (5) are of oscillatory type and weakly decreasing [20, 26]. Correspondingly, numerical solution of the equation (3) has oscillatory character [26] and higher error. Decreasing the step h does not change the character of the solution, and, as it follows from (5), changes only time scale.

Usage of the finite difference method (FDM) with use of approximation of derivative by spatial variable of $O(h^2)$ order for solving the equation (3) results in the following system of differential equations

$$\dot{\tilde{T}}_i = -v\frac{\tilde{T}_{i+1} - \tilde{T}_{i-1}}{2h}, \quad i = \overline{1, n-1}, \tag{6}$$

where $\tilde{T}_i(t)$ is approximate value of the exact solution $T_i(t) = T(t, x_i)$ at the points x_i. Roots of the characteristic equation of the system (6) are pure imaginary, which corresponds to stability boundary of this system. Numerical solution of the system (6) considerably deviates from solution of the equation (3), and for some methods of integration diverges [26].

These negative results of application of numerical methods are connected with the absence of taking into account the character of the solution (3), and, namely, in the considered case (positive velocity) as it follows

from (4) variation of temperature at certain point x_i depends on time only on the looked for function values from the left of this point, as it is shown in Figure 1. This corresponds to physical nature of the considered process, namely, variation of temperature is consequence of medium transfer with the velocity v.

For taking into account this property on application of FDM we approximate the derivative of spatial variable in the equation (3) by the so-called right difference

$$\frac{\partial T(t, x_i)}{\partial x} \approx \frac{T_i - T_{i-1}}{h}, \quad i = \overline{1, n}, \tag{7}$$

or by upstream difference. In this case the system of equations for determination of numerical solution of the equation (3) takes the following form:

$$\dot{\tilde{T}}_i = -v \frac{\tilde{T}_i - \tilde{T}_{i-1}}{h}, \quad i = \overline{1, n}. \tag{8}$$

Roots of the characteristic equation of this system are real and negative, and its solution for the corresponding selection of dimension of grid step coincides with solution of the equation (3) with high accuracy [26].

Similar result may be obtained on application of the Petrov–Galerkin method, which differs from the Galerkin method by usage of weighting functions $W_i(x)$, which do not coincide with the basis ones $N_i(x)$. Application of this method for numerical solution of the equation (3) results in the following equation:

$$\int_0^L \sum_{i=0}^n \dot{a}_i N_i W_j \, dx + v \int_0^L \sum_{i=0}^n a_i \frac{\partial N_i}{\partial x} W_j \, dx = 0. \tag{9}$$

Selection of the functions $W_i(x)$ makes it possible to integrate the expression, which contains derivative $\partial T / \partial x$ or $\partial N_i / \partial x$ with greater weight from incoming flow within support of the weighting function.

In particular, for solving of the stationary equation (1) by the Petrov–Galerkin method it was suggested [20, 28] to use the weighting functions

$$W_i(x) = N_i(x) + \alpha W_i^*(x), \tag{10}$$

where the parameter $\alpha \in [0, 1]$, and the function $W_i^*(x)$ on the interval $[x_{i-1}, x_{i+1}]$ holds the following conditions:

$$W_i^*(x_{i-1}) = W_i^*(x_i) = W_i^*(x_{i+1}) = 0,$$

$$\int_{x_{i-1}}^{x_i} W_i^*(x) \, dx = \int_{x_{i-1}}^{x_i} N_i(x) \, dx = \frac{h}{2}, \quad \int_{x_i}^{x_{i+1}} W_i^*(x) \, dx = -\int_{x_{i-1}}^{x_i} N_i(x) \, dx = -\frac{h}{2}. \tag{11}$$

Outside of the interval $[x_{i-1}, x_{i+1}]$ the function $W_i^*(x)$ is equal to zero identically. For $\alpha = 1$ from (11) and (10) we get $\int_{x_{i-1}}^{x_i} W_i(x) \, dx = h$, $\int_{x_i}^{x_{i+1}} W_i(x) \, dx = 0$. This means that mean value of the weighting function $W_i(x)$ is equal to 1 on interval $[x_{i-1}, x_i]$, i.e., from the direction of incident flow, and is equal to 0 on $[x_i, x_{i+1}]$.

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We can make sure that the following piecewise-polynomial functions hold the condition (11)

$$^{n}W_{i}^{*}(x) = \begin{cases} ^{n}W((x_{i} - x)/h), & x \in [x_{i-1}, x_{i}], \\ -^{n}W((x_{i+1} - x)/h), & x \in [x_{i}, x_{i+1}], \\ 0, & x \notin [x_{i-1}, x_{i+1}], \end{cases} \qquad (12)$$

where the function $^{n}W(\lambda) = [(n+1)/(n-1)]\lambda(1 - \lambda^{n-1})$, n is degree of the polynomial $(n > 1)$. Graphs of functions $N_{i}(x)$ and $W_{i}(x)$ for $\alpha = 1$ and for the values $n = 2;\ 5$ and 100 are shown in Figure 2.
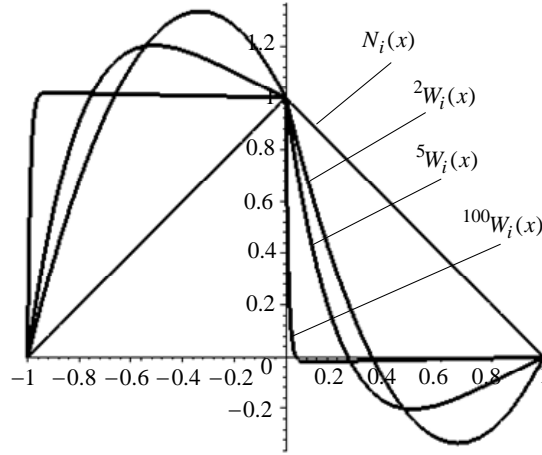
Figure 2

In [28] the functions $^{2}W_{i}^{*}(x)$ were used and the following dependency was obtained

$$\alpha = \coth(Pe \cdot h / 2) - 2/(Pe \cdot h), \qquad (13)$$

for parameter α on the Peklet number $Pe = v / \kappa$, for which in stationary case $(\partial T / \partial t) = 0)$ numerical solution (1) coincides with the exact one at nodes.

We can easily make sure that weighting functions $W_{i}(x)$ of (10) type under the condition $W_{i}^{*}(x) = ^{n}W_{i}^{*}(x)$ and $\alpha = 1$ converge for $n \to \infty$ by the norm $\|\cdot\|_{2}$ [38] to the function equal to 1 on the interval (x_{i-1}, x_{i}) and zero at all other points,

$$W_{i}(x) = \begin{cases} 1, & x \in (x_{i-1}, x_{i}], \\ 0, & x \notin (x_{i-1}, x_{i}]. \end{cases}$$

If we substitute this expression in (9), we obtain the following equations for determination of the expansion coefficients

$$\frac{1}{2}(\dot{a}_{j-1} + \dot{a}_{j}) = -v \frac{a_{j} - a_{j-1}}{h}, \quad j = \overline{1, n}. \qquad (14)$$

This equation practically coincides with the equation (8) and, as it was shown in [26], also makes it possible to obtain numerical solution of the equation (3) with the required accuracy.

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Selection of weighting functions in the Petrov–Galerkin method, which provides the given (required) accuracy of numerical solution, up to date represents unsolved in the general case problem.

Objective of the present article is to show the way of construction of continuous piecewise-polynomial weighting functions, similar to (10), for the Petrov–Galerkin method in two-dimensional domain, to construct by means of this method finite-dimensional model for nonstationary process of convection-diffusion. We assume to verify accuracy of solutions, obtained by means of the constructed model, by comparison with the known analytical solution.

2. Basis and weighting functions in two-dimensional case

We shall suppose that the considered domain $\Omega \subset R^2$ is simply connected, R^2 is two-dimensional real Euclidean space. Components of the vectors $x = (x_1, x_2) \in \Omega$, the numbers x_1 and x_2 are coordinates of points of plane in certain Cartesian coordinate system. We assume that the domain Ω may be represented as finite union of triangular elements (triangles)

$$\Omega = \bigcup_j \Omega_j, \quad \Omega_k \bigcap \operatorname{int} \Omega_l = \varnothing \quad \forall k \neq l, \tag{15}$$

where $\operatorname{int} \Omega_l$ is the set of internal points of a triangular element Ω_l. Only vertexes or completely edges can be common for triangles. As the result all domain will be covered with grid, which consists of edges and vertexes of triangles or nodes. Every vertex is characterized by its number and system of coordinates. For specification of triangular element with the number j it is necessary to specify a set of numbers of its vertexes I_j.

Let us consider piecewise-linear basis functions $N_i(x) = N_i(x_1, x_2)$ [21–24]. Index i of the function $N_i(x)$ means that it is connected with the i-th node. The set of elements Ω_j, to which the i-th node enters forms the polyhedron $\Omega(i)$, shown in Figure 3 (the i-th node is denoted by point in the center).
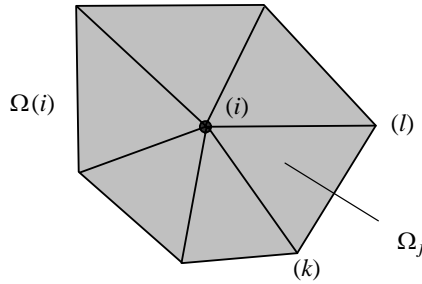

Figure 3

Every basis function $N_i(x)$ is different from zero only inside this polyhedron, is equal to zero on its boundary and to unit at i-th node, $N_i(x_i) = 1$. On every triangular element the function $N_i(x)$ is linear, i.e.,

$$N_i(x) = N_i(x_1, x_2) = a_{ij} + b_{ij} x_1 + c_{ij} x_2, \quad x \in \Omega_j. \tag{16}$$

Here the numbers a_{ij}, b_{ij} and c_{ij} are uniquely determined from the condition that the function $N_i(x)$ is equal to unit at the i-th vertex of triangle Ω_j and to zero at two other ones. For such definition the function $N_i(x)$ is continuous on the polyhedron $\Omega(i)$, as well as in R^2.

Let us state general requirements to the weighting function $W_i(x)$. It should be continuous on the polyhedron of the basis function $N_i(x)$, equal to zero on its boundary and unit at the i-th node, $W_i(x_i) = 1$. Supplementary requirements consist in the fact, that by analogy with one-dimensional case weighting functions $W_i(x)$ should have greater weight from the direction of incoming flow.

We assume that on every element Ω_j of components of the polyhedron $\Omega(i)$ the function $W_i(x)$ is defined by polynomial of n dimensionality and, therefore, is continuous on Ω_j. For continuity of the function $W_i(x)$ on the whole polyhedron $\Omega(i)$ its continuity on edges of adjacent elements is necessary and sufficient. Coordinates of points on the edge $(i - k)$, which connects i- and k-th nodes, are determined by the expression

$$x = x(\lambda) = (1-\lambda)x_i + \lambda x_k, \ \ \lambda \in [0, 1]. \tag{17}$$

If we substitute this equality into expression for the polynomial $W_i(x)$, we can make sure that on the edge $(i - k)$ this function is also polynomial of n degree, but of one variable λ. For continuous function $W_i(x)$ on the edge $(i - k)$ it is necessary the polynomials of two adjacent elements on this edge coincide. Let us demand that on every edge $(i - k)$ the function $W_i(\lambda) = W_i(x(\lambda))$, which is considered as function of the parameter λ, is defined by the expression:

$$W_i(\lambda) = N_i(\lambda) + \alpha_{i,k} {}^nW_i^{\text{sign}\,\alpha_{i,k}}(\lambda). \tag{18}$$

Here $\alpha_{i,k}$ is adjusting numerical parameter, connected with the edge $(i - k)$, $\alpha_{i,k} \in [-1, 1]$, $N_i(\lambda) = N_i(x(\lambda)) = 1 - \lambda$. The function $W_i^{\text{sign}\,\alpha_{i,k}}(\lambda)$ for different values of the number α_{ik} is defined by the following expressions:

$$ {}^nW_i^-(\lambda) = -{}^nW(\lambda), \ \ {}^nW_i^+(\lambda) = -{}^nW(1-\lambda), \ \ \lambda \in [0, 1]. \tag{19}$$

For $\alpha_{i,k} = -1$, as it follows from (18), (19), the function $W_i(\lambda)$ coincides with the function (10) on the interval $[x_{i-1}, x_i]$. If $\alpha_{i,k} = +1$, then the function $W_i(\lambda)$ coincides with (10) on the interval $[x_i, x_{i+1}]$. For $\alpha_{i,k} = 0$ we have $W_i(\lambda) = N_i(\lambda)$. Thus, by means of selection of the parameter $\alpha_{i,k}$ we can set different form of graph of the function $W_i(x)$ on every edge $(i - k)$ of the element $\Omega(i)$.

Let us consider construction of the function $W_i(x)$ for $n = 2$. In this case the function $W_i(x)$ on the set Ω_j is given by the polynomial

$$W_{ij}(x) = W_{ij}(x_1, x_2) = a_{ij} + b_{ij}x_1 + c_{ij}x_2 + d_{ij}x_1x_2 + f_{ij}x_1^2 + g_{ij}x_2^2. \tag{20}$$

Triangular element Ω_j contains the vertex with the number i, we denote two other vertexes as k and l. According to requirements to the function $W_i(x)$ the values of $W_{ij}(x)$ on the edges $(i-k)$ and $(i-l)$ are defined by the expressions (18) and (19), and on the edge $(k-l)$ they are zero. For determination of six unknown coefficients of the function $W_{ij}(x)$ we specify its values in six nodes of the element Ω_j. Besides nodes in vertexes of the triangular element Ω_j we select supplementary nodes in the middle of its sides, for example, as it is shown by points in Figure 4. We shall denote by index i and k points in the middle of side, which connects, for example, nodes i and k, i.e., we write $x_{i,k} = (x_i + x_k)/2$. As the result we obtain the following six equalities:

$$W_{ij}(x_i) = 1, \ W_{ij}(x_k) = W_{ij}(x_l) = W_{ij}(x_{k,l}) = 0,$$

$$W_{ij}(x_{i,k}) = 0.5 - 0.75\,\alpha_{i,k}, \ W_{ij}(x_{i,l}) = 0.5 - 0.75\,\alpha_{i,l}. \tag{21}$$

On obtaining two last equalities we took into account that $^2W_i^-(0.5) = \,^2W_i^+(0.5) = -\,^nW(0.5) = -3 \cdot 0.25 = \,= -0.75$. Using the equalities (21) and the expression (22) for determination of the coefficients a, b, c, d, f and g (indexes ij are omitted) we obtain the system of linear algebraic equations

$$\begin{pmatrix} 1 & x_{i1} & x_{i2} & x_{i1}x_{i2} & x_{i1}^2 & x_{i2}^2 \\ 1 & x_{k1} & x_{k2} & x_{k1}x_{k2} & x_{k1}^2 & x_{k2}^2 \\ 1 & x_{l1} & x_{l2} & x_{l1}x_{l2} & x_{l1}^2 & x_{l2}^2 \\ 1 & x_{k,l1} & x_{k,l2} & x_{k,l1}x_{k,l2} & x_{k,l1}^2 & x_{k,l2}^2 \\ 1 & x_{i,k1} & x_{i,k2} & x_{i,k1}x_{i,k2} & x_{i,k1}^2 & x_{i,k2}^2 \\ 1 & x_{i,l1} & x_{i,l2} & x_{i,l1}x_{i,j2} & x_{i,l1}^2 & x_{i,l2}^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ f \\ g \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0.5 - 0.75\,\alpha_{ik} \\ 0.5 - 0.75\,\alpha_{il} \end{pmatrix}. \tag{22}$$

By means of parameters $\alpha_{i,k}$ and $\alpha_{i,l}$ we can change the form of graph of the function $W_{ij}(x)$: for $\alpha_{i,k}$, $\alpha_{i,l} > 0$ and make it convex, as it is shown in Figure 5, for $\alpha_{i,k}, \alpha_{i,l} < 0$ we can do it concave (Figure 6) or convex-concave for $\alpha_{i,k} \cdot \alpha_{i,l} < 0$ (Figure 7)
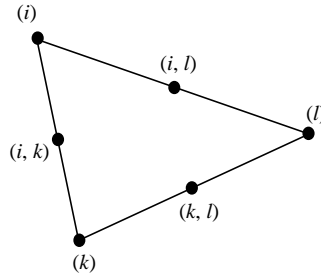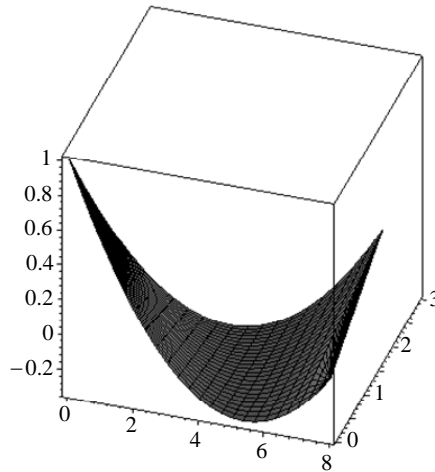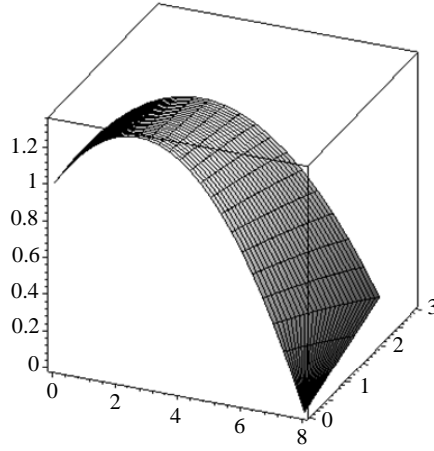

Figure 4
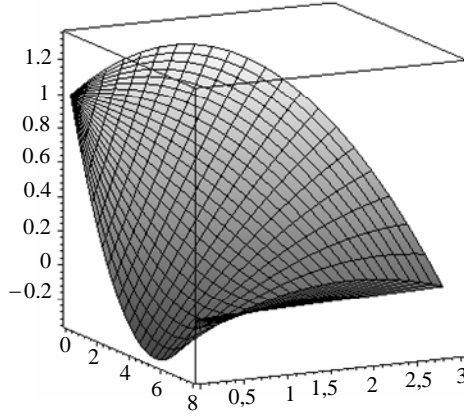

Figure 5

Figure 6

Figure 7

We write the equation (22) in the vector-matrix form

$$Xp = h_0 + \alpha_{i,k} h_k + \alpha_{i,l} h_l, \tag{23}$$

where X is the matrix in the expression (22), the vectors $p = (a, b, c, d, f, g)^{\mathrm{T}}$, $h_0 = (1, 0, 0, 0, 0.5, 0.5)^{\mathrm{T}}$, $h_k = (0, 0, 0, 0, -0.75, 0)^{\mathrm{T}}$, $h_l = (0, 0, 0, 0, 0, -0.75)^{\mathrm{T}}$. We obtain from (23)

$$p = X^{-1} h_0 + \alpha_{i,k} X^{-1} h_k + \alpha_{i,l} X^{-1} h_l \tag{24}$$

or

$$p = p_0 + \alpha_{i,k} p_k + \alpha_{i,l} p_l, \tag{25}$$

where the vector $p_0 = X^{-1} h_0$ contains coefficients of the function $N_{ij}(x)$, and the vectors $p_k = X^{-1} h_k$ and $p_l = X^{-1} h_l$ contain coefficients of certain polynomials $W_{(i,k)j}(x)$ and $W_{(i,l)j}(x)$, correspondingly. Therefore, the following equality for functions corresponds to the equality (25)

$$W_{ij}(x) = N_{ij}(x) + \alpha_{i,k} W_{(i,k)j}(x) + \alpha_{i,l} W_{(i,l)j}(x). \tag{26}$$

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The polynomial $W_{(i,k)j}(x)$ differs from zero on the edge $(i-k)$ and is identically equal to zero on edges $(i-l)$ and $(k-l)$ of the element Ω_j. On the element $\Omega_{j'}$ adjacent to the element Ω_j by the edge $(i-k)$ the functions $W_{(i,k)j'}(x)$ possess similar properties. Let us consider the function

$$W_{(i,k)}(x) = \begin{cases} W_{(i,k)j}(x), & x \in \Omega_j, \\ W_{(i,k)j'}(x), & x \in \Omega_{j'}, \end{cases} \tag{27}$$

defined on union of adjacent elements $\Omega_j \bigcup \Omega_{j'}$. The function $W_{(i,k)}(x)$ is continuous and is equal to zero everywhere, except the set $\Omega_j \bigcup \Omega_{j'}$, in the middle of the edge $(i-k)$, as it follows from definition, its value is -0.75. We can define similar for all edges, which come from the node with the number i.

As the result the weighting function $W_i(x)$ on the element $\Omega(i)$ is representable as

$$W_i(x) = N_i(x) + \sum_{k \in K_i} \alpha_{i,k} W_{(i,k)}(x). \tag{28}$$

Here K_i is the set of vertex numbers, which are connected with the vertex i. In practice the number of edges, coming from certain node i seldom exceeds the number 6. Every of the functions $N_i(x)$ and $W_{(i,k)}(x)$ and, therefore, $W_i(x)$ is specified by its separate expression on every of elements Ω_j, which form the set $\Omega(i)$.

3. Finite-dimensional model for two-dimensional nonstationary process of convection–diffusion

We write the heat transfer equation in the case of two spatial variables in the following way:

$$\frac{\partial T}{\partial t} + v \cdot \nabla T = \kappa \Delta T, \ T = T(t, x), \ x \in \Omega, \ t \in [t_0, t_1], \tag{29}$$

where $\nabla T = (\partial T / \partial x_1, \partial T / \partial x_2)$ is gradient of the temperature field $T = T(t, x)$, $\Delta \partial^2 / \partial x_1^2 + \partial^2 / \partial x_2^2$ is the Laplace operator. In the equation (29) the field of velocities $v(t, x) = (v_1(t, x), v_2(t, x))$ is assumed to be everywhere smooth in the domain Ω, $v \cdot \nabla T = v_1 \partial T / \partial x_1 + v_2 \partial T / \partial x_2$ is scalar product of the vectors v and ∇T. We assume that on the boundary of the domain Ω one of standard boundary conditions are stated [12]. At the place, where this is essential, it is necessary to suppose that the conditions of the 1-st kind are stated.

We shall look for the approximate weak solution [9, 36] of this equation as

$$\hat{T}(t, x) = \sum_{i=1}^{n} a_i(t) N_i(x). \tag{30}$$

Here $N_i(x)$ is piecewise-linear basis function corresponding to i-th grid node. Taking into account that the vector of velocity may be solution of FEM hydrodynamic equations, we represent it also in the form of decomposition by basis functions

$$v(t, x) \cong \hat{v}(t, x) = \sum_{i=1}^{n} V_i(t) N_i(x), \tag{31}$$

where the vector $V_i(t) = (v_1(t, x_i), v_2(t, x_i))$. According to formal procedure of the Petrov–Galerkin method we substitute the expressions (30) into the equation (29), multiply the obtained equality by the

weighting function $W_j(x)$, which is defined by (28), and integrate over the domain Ω. Taking into account the expression (31) we have

$$\sum_{i=1}^{n}\int_{\Omega} N_i W_j \, d\Omega \, \dot{a}_i + \sum_{i=1}^{n}\sum_{l=1}^{n}\left(\int_{\Omega}\frac{\partial N_i}{\partial x_1}N_l W_j \, d\Omega\right)V_{1l}a_i + \sum_{i=1}^{n}\sum_{l=1}^{n}\left(\int_{\Omega}\frac{\partial N_i}{\partial x_2}N_l W_j \, d\Omega\right)V_{2l}a_i =$$

$$= \kappa\sum_{i=1}^{n}\left[\left(\int_{\Omega}\frac{\partial N_i}{\partial x_1}\frac{\partial W_j}{\partial x_1}\, d\Omega\right)+\left(\int_{\Omega}\frac{\partial N_i}{\partial x_2}\frac{\partial W_j}{\partial x_2}\, d\Omega\right)\right]a_i + f_j. \qquad (32)$$

For obtaining this equation we used integration by parts from expressions, which contain the second derivatives by spatial variables. Taking into account boundary conditions the value f_j is connected [9, 21–23].

Starting from denotations

$$D^{ij} = \int_{\Omega} N_i W_j \, d\Omega, \ \ D_{11}^{ij} = \int_{\Omega}\frac{\partial N_i}{\partial x_1}\frac{\partial W_j}{\partial x_1}\, d\Omega, \ \ D_{22}^{ij} = \int_{\Omega}\frac{\partial N_i}{\partial x_2}\frac{\partial W_j}{\partial x_2}\, d\Omega,$$

$$T_1^{ilj} = \int_{\Omega}\frac{\partial N_i}{\partial x_1}N_l W_j \, d\Omega, \ \ T_2^{ilj} = \int_{\Omega}\frac{\partial N_i}{\partial x_2}N_l W_j \, d\Omega, \qquad (33)$$

we rewrite the equation (32) as

$$\sum_{i=1}^{n}D^{ij}\dot{a}_i + \sum_{i=1}^{n}\sum_{l=1}^{n}T_1^{ilj}V_{1l}a_i + \sum_{i=1}^{n}\sum_{l=1}^{n}T_2^{ilj}V_{2l}a_i = \kappa\sum_{i=1}^{n}[D_{11}^{ij}+D_{22}^{ij}]a_i + f^{\,j}, \ \ j\in J_0, \qquad (34)$$

where J_0 is the set of numbers of internal nodes of partition grid. If we substitute (28) into expression for the coefficients $D_{11}^{ij}+D_{22}^{ij}$, we get

$$D_{11}^{ij}+D_{22}^{ij} = \int_{\Omega}\frac{\partial N_i}{\partial x_1}\frac{\partial W_j}{\partial x_1}\, d\Omega + \int_{\Omega}\frac{\partial N_i}{\partial x_2}\frac{\partial W_j}{\partial x_2}\, d\Omega =$$

$$= \int_{\Omega}\frac{\partial N_i}{\partial x_1}\left(\frac{\partial N_j}{\partial x_1}+\sum_{k\in K_j}\alpha_{j,k}\frac{\partial W_{(j,k)}}{\partial x_1}\right)d\Omega + \int_{\Omega}\frac{\partial N_i}{\partial x_2}\left(\frac{\partial N_j}{\partial x_2}+\sum_{k\in K_j}\alpha_{j,k}\frac{\partial W_{(j,k)}}{\partial x_2}\right)d\Omega =$$

$$= \int_{\Omega}\frac{\partial N_i}{\partial x_1}\frac{\partial N_j}{\partial x_1}\, d\Omega + \int_{\Omega}\frac{\partial N_i}{\partial x_2}\frac{\partial N_j}{\partial x_2}\, d\Omega + \sum_{k\in K_j}\alpha_{j,k}\left(\int_{\Omega}\frac{\partial N_i}{\partial x_1}\frac{\partial W_{(j,k)}}{\partial x_1}\, d\Omega + \int_{\Omega}\frac{\partial N_i}{\partial x_2}\frac{\partial W_{(j,k)}}{\partial x_2}\, d\Omega\right). \qquad (35)$$

We connect the coefficients $\alpha_{j,k}, \ k\in K_j$, by analogy with one-dimensional case with the value of the vector of velocity $V_j(t)=v(t,x_j)$ in the j-th node at time instant t, and namely, it is necessary to camber every function upward from the side of incoming flow.

Let us consider several variants of heuristic selection of the parameters $\alpha_{j,k}$.

Variant 1:

$$\alpha_{j,k} = \alpha(Pe)V_j\cdot\Delta x_{kj}/(\|V_j\|\cdot\|\Delta x_{kj}\|), \ \ \forall k\in K_j, \qquad (36)$$

where $\Delta x_{kj}=x_k-x_j$, $\|x\|=(x\cdot x)^{1/2}$ is norm of the vector x, the function $\alpha=\alpha(Pe)$ is defined by (13).

Variant 2:

$$\alpha_{j,k} = \alpha(Pe)\,\text{sign}\,(V_j \cdot \Delta x_{kj}), \quad \forall k \in K_j. \tag{37}$$

Variant 3;

$$\alpha_{j,k} = \alpha(Pe)F(V_j \cdot \Delta x_{kj} /(\|V_j\| \cdot \|\Delta x_{kj}\|)), \quad \forall k \in K_j, \tag{38}$$

where the function $F: R^1 \to R^1$ is defined by the expression

$$F(z) = \begin{cases} \beta z, & |\beta z| \le 1, \\ \text{sign}\,(\beta z), & |\beta z| > 1. \end{cases} \tag{39}$$

Here β is certain number, $\beta \ge 0$. For $\beta = 0$ we obtain the Galerkin method, where weighting functions coincide with basis ones; for $\beta = 1$ variant 3 transfers to variant 1, and for $\beta = \infty$ it transfer into variant 2. The parameter β makes it possible to change quickly parameters of the equation (34).

It is possible to select the coefficients $\alpha_{j,k}$ in such a way that

$$\sum_{k \in K_j} \alpha_{j,k} \left(\int_\Omega \frac{\partial N_i}{\partial x_1} \frac{\partial W_{(j,k)}}{\partial x_1} d\Omega + \int_\Omega \frac{\partial N_i}{\partial x_2} \frac{\partial W_{(j,k)}}{\partial x_2} d\Omega \right) = 0. \tag{40}$$

In this case the sum of the right-hand part of the equations (34) coincides with expression, obtained by the Galerkin method. Namely this takes place on application of the Petrov–Galerkin method in one-dimensional case. Let us denote the number of elements of the set K_j by the symbol K and elements of this set by $k_1, \ldots, k_K$. Then the system of coefficients $\alpha_{j,k}$, $k \in K_j$, can be considered as the vector $\alpha_j = (\alpha_{j,k_1}, \ldots, \alpha_{j,k_K})^\text{T} \in R^K$. Here the condition (40) defines certain hyperplane in the space R^K, which passes through the origin. Supplementary restrictions for the coefficients $\alpha_{j,k}$

$$-1 \le \alpha_{j,k} \le 1, \quad k \in K_j, \tag{41}$$

specify multidimensional cube in R^K. Let us denote intersection of this cube with the hyperplane, defined by the equality (40), by A_j. This set represents polyhedron in R^K. For realization of (40) we project the vector of coefficients α_j, calculated in accordance with one of the mentioned above variants, on the set A_j. Introduction of the condition (40) makes it possible to simplify additionally the system of equations (34).

4. Accuracy of the model

Let us consider boundary value problem for the equation (29) in rectangular domain $0 \le x_i \le L_i$, $i = 1, 2$, for constant vector of velocity $v = (v_1, v_2)^\text{T} = \text{const}$. The initial condition is specified in the form $T(0, x_1, x_2) = e^{x_1}$. Boundary conditions are prolongation by continuity of initial conditions on boundary of the domain $T(t, 0, x_2) = 1$, $T(t, L_1, x_2) = e^{L_1}$, $T(t, x_1, 0) = T(t, x_1, L_2) = e^{x_1}$.

We obtain analytical solution of the problem for its further comparison with numerical solution. If we use substitute in the form

$$T(t, x_1, x_2) = e^{at + bx_1 + cx_2} u(t, x_1, x_2), \tag{42}$$

where

$$a = -(v_1^2 + v_2^2)/4\kappa = -\|v\|^2/4\kappa, \quad b = v_1/2\kappa, \quad c = v_2/2\kappa, \tag{43}$$

the equation (29) is reduced to the diffusion equation for the function $u = u(t, x_1, x_2)$, which is solved by the method of separation of variables [12]. Solving the initial boundary value problem for the equation (29) with the stated above initial and boundary conditions is expresses as

$$T(t, x_1, x_2) = e^{at+bx_1+cx_2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{H_{ij}}{\Omega_{ij} - a}(e^{-at} - e^{-\Omega_{ij}t})\sin i\frac{\pi x_1}{L_1}\sin j\frac{\pi x_2}{L_2} + e^{x_1}, \qquad (44)$$

where

$$\Omega_{ij} = \kappa\left(\left(\frac{\pi i}{L_1}\right)^2 + \left(\frac{\pi j}{L_2}\right)^2\right),$$

$$H_{ij} = 4\pi^2 ij(\kappa(1-b)^2 + \kappa c^2 + a)\frac{1 + e^{L_1(1-b)}(-1)^{i+1}}{L_1^2(1-b)^2 + \pi^2 i^2} \cdot \frac{1 + e^{-L_2 c}(-1)^{j+1}}{L_2^2 c^2 + \pi^2 j^2}.$$

Numerical solution of the considered problem is found by the classical Galerkin method $(\alpha_{i,k} \equiv 0 \ \forall i,k)$, and by the Petrov–Galerkin method, where parameters specifying the form of weighting functions, were computed according to (36)–(38). Here we used the following numerical parameters of the problem $v_1 = v_2 = 50$, $\kappa = 0.1$, $L_1 = 1$, $L_2 = 1$. It is seen, that for such ratio of the vector of velocity v and the coefficient κ the Peclet number is $Pe \approx 700$, i.e., in the considered problem the convection processes dominate diffusion process. The result of numerical solving of the problem for $y = 0.5$ by the Galerkin method for uniform partition of the domain into 15×15 nodes is represented in Figure 8.
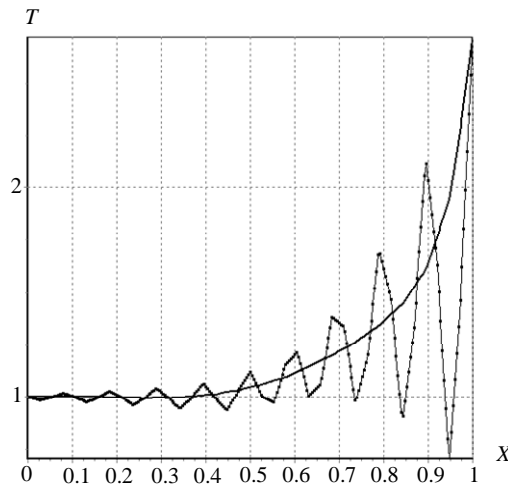

Figure 8

In Figure 9 we show the graph of the solution, which is obtained by the Petrov–Galerkin method on use of variant 1 for selection of parameters of weighting functions.

Better in comparison with previous case result was obtained on selection of parameters of weighting functions according to variant 3 for $\beta = 1.5$. The corresponding graph is shown in Figure 10.

Variant 2 of selection of parameters of the function turned out to be the most exact. Graph of solution is shown in Figure 11.

Reduction of step value by spatial variable in all cases makes it possible to increase accuracy of numerical solution.
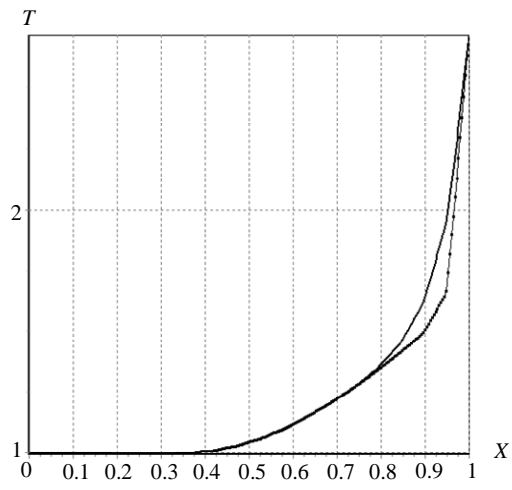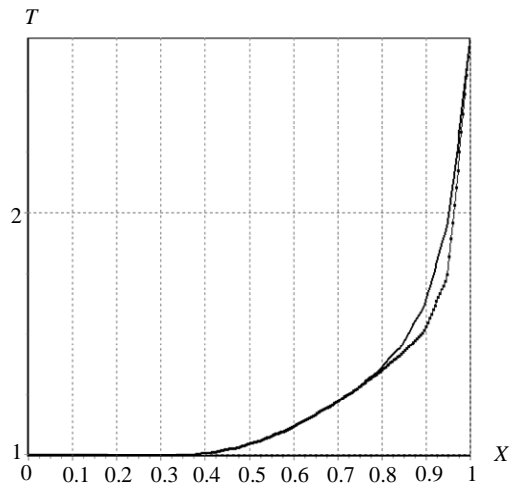
T

2

1
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
X

Figure 9

T

2

1
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
X

Figure 10

T

2

1
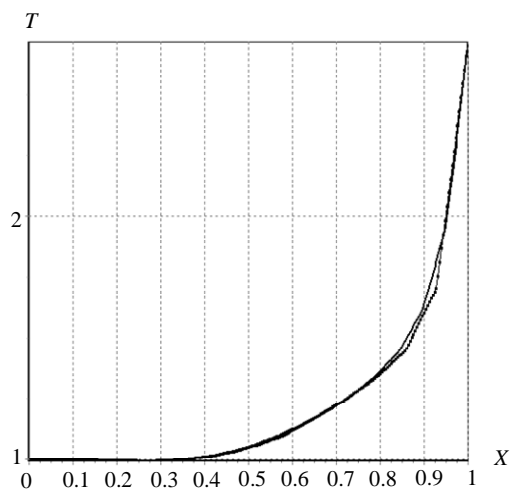0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
X

Figure 11

81

Conclusion

In the present article we solved the problem of construction of finite-dimensional model of the process of convection-diffusion in two-dimensional domain of arbitrary shape with usage of the Petrov–Galerkin method. Here we suggested a way of construction of continuous piecewise-polynomial weighting function for the Petrov–Galerkin finite element method, which represents generalization of one-dimensional weighting function [20, 28] for two-dimensional case. Weighting function is specified for every node by means of independent parameters, connected with grid segments, which contain the given node. Efficiency of selection of parameters was determined by comparison of numerical solution with testing analytical one. All variants of selection provided good qualitative coincidence and stability of mathematical model, obtained by means of the Petrov–Galerkin method. The model, obtained by the Galerkin method turned out to be unstable. Usage of piecewise-polynomial function $W_i(x)$ makes it possible to realize analytical integration on an element of expressions, which contain product of the functions $N_i(x)$ and $W_i(x)$ and/or their derivatives, by known formulae [23]. This speeds up considerably determination of parameters of the system of differential equations (34) in comparison with numerical integration of these expressions. This is of importance if parameters of the equation (34), for example, vector of velocity, are nonstationary.

It is necessary to underline universality of the suggested method of construction of finite-dimensional mathematical model of process of convection-diffusion for domains of complex shape and potential of automation of the whole process of model obtaining.

It is possible to transfer the suggested technique of construction of weighting functions without any complexities for three-dimensional case. Selection of values of parameters, which characterize the form of the weighting function $W_i(x)$, optimal relative to accuracy of the obtained model, remains to be the subject of further research. Usage of the suggested weighting functions for construction of finite-dimensional approximations of other equations of mathematical physics, containing convective terms, is of interest, in particular, for solving the Navier–Stocks equations, as well as for equations of magnetic hydrodynamics.

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