Mechanics of Liquids and Gases

6th edition

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Library of Congress Cataloging-in-Publication Data

Loitsyanskiy , L.G. (Lev Gerasimovich)
(Mekhanika zhidkosti I gaza. English)
Mechanics of liquids and gases /L.G. Loitsyanskiy.-- 6th ed
p. cm.
Includes indexes.
ISBN 1-56700-042-8 (hardcover)
1. Hydrodynamics. 2. Aerodynamics. I. Title.
QA911.L613 1995
532'.05--dc20 95-25679
CIP

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Author's Preface

This publication is the translation of the 6th edition (1987, Nauka Press, Moscow) of my monographical textbook "Mechanics of Liquids and Gases" which was extensively revised and augmented as compared with the 2nd edition (1957, Nauka Press, Moscow) translated into English and published by Pergamon Press in 1966. The 6th edition incorporates significant advances towards the solution of the problems of the boundary layer theory, dynamics of viscous fluids and theory of turbulence. Moreover, this English edition is augmented with the 16th Chapter on computational hydrogasdynamics written by my colleagues at the Department of Hydroaerodynamics of the Leningrad Polytechnic Institute: Prof. Yu. P. Golovachev and Ass. Prof. S. B. Koleshko to whom, taking this opportunity, I wish to express my gratitude. I also wish to thank the Joint Soviet-American Enterprize "TEMPO" in the persons of its founders Mr. W. Begell and Corresponding Member of the Byelorussian Academy of Sciences O. G. Martynenko and also staff-members of the USSR Editorial Office of the International Journal of Heat and Mass Transfer G. R. Malyavskaya and N. K. Shveyeva who has taken the trouble to translate my book into English.

The textbook is intended for students, postgraduate students, engineers, and research workers specializing in the field of mechanics.

Author's Biography

Born on 26 December 1900 in St. Petersburg, L. G. Loitsyanskiy graduated in 1921 from the Physico-Mathematical Department of the University with the speciality "pure mathematics". From 1922 to 1934, L. G. Loitsyanskiy is Assistant Lecturer, then Assistant Professor, thereafter Professor at the Department of Theoretical Mechanics of the Petrograd (Leningrad) Polytechnical Institute and thereupon he accepted the chairmanship of the Hydroaerodynamic Department of the same Institute.

L. G. Loitsyanskiy is Doctor of Physical and Mathematical Sciences (1935), State Prize Winner (1946), Merited Worker of Science and Technology (1968), Full Member of the International Astronautical Academy (1970), Foreign Member of the Polish Mathematical Society (1972), Honored Member of the Yugoslavian Society of Mechanics (1982).

L. G. Loitsyanskiy has authored more than 120 scientific works in the fields of applied mathematics, dynamics of viscous fluids, theory of boundary layer and turbulence. Among these works are textbooks, monographs and scientific reviews, *viz.* "Theoretical Mechanics" in 3 volumes (1933) and "A Course in Theoretical Mechanics" in 2 volumes (1934) co-authored with A. I. Luriye (Corresponding Member of the USSR Academy of Sciences); "Fundamentals of the Mechanics of Viscous Fluids" (in two parts, 1933), "Aerodynamics of the Boundary Layer" (1941), "Laminar Boundary Layer" (1962), "Mechanics of Liquids and Gases" (1950), a review "The Development of Boundary Layer Theory in the USSR" (1970, Ann. Review of Fluid Mechanics, Palo Alto, California, USA).

Preface to the 6th Edition

The present edition is the outcome of an extensive revision made, on the one hand, to bring its content closer to that of a textbook and, on the other, to update it by adding some new present-day problems.

Since it was impossible to extend any further so bulky a book as it was, some special sections dealing with the mechanics of liquids and gases but not fulfilling the objectives of the present general consideration had to be sacrificed. Thus, the problem of motion of multiphase media was dropped from the dynamics of inhomogeneous fluids, and only the derivation of the equations for multicomponent gas flows was retained as being closely associated with the description of the phenomena of dissociation and ionization of molecules in supersonic gas flows. Also omitted was the section dealing with the macroscopic approach to calculations of dust-laden gas flows which gave way to the methods of the kinetic theory of molecular motions. This was also the reason for dropping the section which discussed the shock wave from the viewpoint of continuum mechanics as well as some other problems dealt with in the previous edition of the book.

As distinct from previous editions where only vector and tensor calculus formulae were given, the present one provides a short background discussion on this area of mathematics. The section on the elements of similarity theory is supplemented with a presentation of dimensionality theory and with the proof of the main theorem in this theory.

Newly written are three sections dealing with some general methods for numerical integration of differential equations and their application to Navier-Stokes viscous fluid dynamics equations. Extensive revision of the chapter on turbulence culminated in the appearance of a new chapter dealing specially with the techniques for calculating turbulent boundary layers. Methodological blunders noticed in many places throughout the book have been eliminated.

Professor G. Yu. Stepanov, Dr. Sc. (Physics and Mathematics) read the manuscript with an exceptional accuracy and made a great number of scientific and methodological comments important for the quality of the book.

Professor Yu. V. Lapin, Dr. Sc. (Physics and Mathematics) read the manuscript of Chapter XIV and made a number of useful comments. An invaluable help in the work on the manuscript and in proof-reading was lent by Assistant Professor at the Chair of Hydroaerodynamics of Leningrad Polytechnic Institute S. B. Koleshko who not only undertook the scientific editing of the manuscript and made many useful remarks, but also wrote three sections (102–104) concerned with numerical solution of viscous fluid dynamics problems. The author regards it his duty to acknowledge cordially their assistance for this.

Preface to the English Edition

Many of us who study and teach "fluid mechanics" have come to think of it as some sort of counterpart, or alternative, to "solid mechanics." Indeed, western university curricula typically address the two subjects with specialized faculty using texts that do little to reveal their common ground. Fluid mechanics to Loitsyanskiy, as his title proclaims, springs naturally from the vast realm of mechanics itself. Thus his technique is to use, over and over again, the general formulations of mechanics as the foundation of theories describing the behaviors of fluids--viscous and inviscid, liquids and gases.

The result may be somewhat daunting to the pragmatic student of fluid mechanics, for the practical results are often achieved by Loitsyanskiy only after substantial mathematics gyrations of the most general kind. Readers should take heart, however, because if the material seems abstract there are often repeated explanations with very helpful nuances. For the fluid mechanics devotee (especially non-Russian readers, I suspect) Loitsyanskiy offers words that are exquisitely chosen; and many new ways of looking at the wonderful world of fluids are waiting for those who persevere. As one of many examples, I recall how airfoil theory is almost completely expounded by Loitsyanskiy without even defining the famous Joukowsky transformation.

As the editor of the translation, I must compliment the translators whose work makes it clear that they have been alert to the significance of the document and the danger of losing even the smallest part of its "personality" in translation. I, also, have done my best not to interfere. Occasionally, I have even left some "fuzzy" passages in their as translated form for fear of losing the beauty of the narrative for the sake of technical precision. In these instances, readers will have to reach their own conclusions.

To be asked to edit the English translation of *Mechanics of Liquids and Gases* was, to me, like being invited to a private viewing prior to the unveiling of a great masterpiece. *The Mechanics of Liquids and Gases* is huge--in coverage and in quality--and its availability in English is a great new learning opportunity.

Robert H. Nunn, Ph.D.
formerly with the US Naval Postgraduate School
Monterey, California

the mechanics of fluids and ensuing general integrals or theorems. In order application of these equations may be fully definitive, it is necessary that the following additional *phenomenological laws* be taken into account: state equation, Newton's law for viscosity, Fourier's law for heat conduction, Fick's law for diffusion, etc.

2. Some Information on the Molecular Structure of Substance

Depending on the quantitative relationship between the kinetic energy of the motion of molecules and potential energy of intermolecular dynamic interaction, there originate different molecular structures and types of internal motion of molecules.

Essential in *solid bodies* is the molecular energy of the interaction of molecules due to which the latter are arranged in regular crystal lattices with stable equilibrium positions at lattice nodes. Thermal motions in a solid body consist of small vibrations of molecules about lattice nodes with a high frequency (of order 10^{12} Hz) and amplitude proportional to spacings between these nodes. Both "short-range" and "long-range" orders are effected in the molecular structure of a solid body. To solid bodies also belong substances in amorphous state which do not have crystalline structure but which possess "short-range" order closely resembling that in liquids (see below). Amorphous states are not very stable and change easily to crystalline states.

In contrast to a solid body, both "short-range" and "long-range" orders are absent in gases. The molecules of a gas move in a random motion, with their interaction being reduced only to collisions. The interaction of molecules in the intervals between collisions is neglected, and this corresponds to the smallness of the potential energy of the dynamic interaction of molecules as compared with the kinetic energy of their random motion. The mean distance between two consecutive collisions of molecules determines the "free path length". The "free path" velocity of molecules is commensurable with the speed of propagation of small disturbances (speed of sound) in this state of gas.

As to their molecular structure and thermal motion, liquid bodies occupy an intermediate position between solid and gaseous bodies. According to the current views, a certain molecule, acting as a central one, collects around itself a group of neighboring molecules that slightly vibrate with a frequency close to that mentioned earlier for vibrations of solid body molecules in a lattice and with an amplitude of the order of mean distance between molecules. The central molecule either remains stationary (in a liquid at rest), or migrates at a speed coinciding in magnitude and direction with the local mean velocity of the macroscopic motion of liquid. In the molecular structure of a liquid the potential energy of molecular interaction is comparable in order with the kinetic energy of their thermal motion, with the "short-range" order being present and the "long-range" order not. The evidence for the vibrations of molecules in liquids is provided by the well-known "Brownian motion" of tiny solid particles introduced into liquid. Vibrations of these particles can be easily seen under a microscope and may be looked upon as the result of collision of solid particles with the molecules of liquid.

The difference between molecular structures and thermal motions of solid, liquid and gaseous bodies reveals itself clearly in the phenomenon of diffusion consisting in the propagation of one substance (inclusion) into the other (carrier). Diffusion of one gas in another (for example, the propagation of odor in air) due to intensive molecular motion leads to rapid penetration of odor into the farthest corners in a room. Conversely, the diffusion of a liquid in a liquid occurs much more slowly because of weak migration of central molecules with groups of molecules bound around them. The best example is provided by the historical experiment of Reynolds who introduced a thin jet of dye into a (laminar) water flow slowly moving through a cylindrical pipe. The jet remained nearly the same in thickness over almost

the mechanics of fluids and ensuing general integrals or theorems. In order application of these equations may be fully definitive, it is necessary that the following additional *phenomenological laws* be taken into account: state equation, Newton's law for viscosity, Fourier's law for heat conduction, Fick's law for diffusion, etc.

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the entire working section of the pipe, thus testifying to a slow molecular diffusion typical of laminar motion of water. On transition to a turbulent mode of flow, when the molecular mechanism of diffusion gives place to turbulent mixing of finite macrovolumes of liquid, there arises an intensive turbulent diffusion and the dye rapidly fills the entire flow.

It should be noted that the phenomenon of diffusion is also observed in solid bodies, although it is much weaker here than in liquids. Mutual penetration of molecules can be seen in two specimens of different metals tightly fitted to each other after the lapse of a long period.

The problems of astrophysics associated with the study of ionosphere, "star clouds" and other astronomical objects and especially various physico-technical problems connected with the design of thermonuclear reactors and magnetohydrodynamic generators for direct conversion of thermal into electrical energy spurred a considerable upsurge of interest in the dynamics of ionized gases (plasmas).

In contrast to ordinary electrically neutral gases in which randomly moving molecules display dynamic interaction only ion only on their mutual collision, much more substantial Coulomb interactions originate in plasma due to a high concentration of charged particles. This imparts specific properties to plasma, as e.g. higher electrical conductivity, showing up most vividly on exposure of plasma flows external electrical and magnetic fields.

The exceptional, in their physical importance and applied possibilities, and at the same time very specific properties of plasmas suggested the idea of the fourth (after solid, liquid and gaseous) aggregated state of matter.[1]

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[1] A detailed survey of the properties of plasma and phenomena occurring in it can be found in the paper by B. B. Kadomtsev "Plasma" (*Physical Encyclopedic Dictionary*, Vol. 4, Moscow, Sov. Entsiklopediya Press, 1965, pp. 15–24) and also in Sects. 1.3–1.4 of the book by Lukiyanov G. A. *Sverkhzvukovye strui plazmy* (*Supersonic Plasma Jets*). Leningrad, Mashinostroeniye Press, 1985, pp. 13–21 and the list of references in it.

1

The Field of a Physical Quantity and Conditions for the Physical Objectivity of Quantities Specified Analytically: Basic Operations of the Field

1. Scalar and Vector Fields: Conditions for the Physical Objectivity of Scalar and Vector Quantities Specified Analytically

The *field* of a physical quantity (velocity, acceleration, pressure, density, temperature, etc.) is the name given to a set of the values of this quantity unambiguously determined at each point in some part of space. There may exist *scalar*, *vector*, and, as seen later, *tensor fields*, depending on the kind of the quantity, the distribution of which is prescribed by the field.

If scalar, vector, or tensor quantities have the same values at all the points of the field, the field is called *uniform*; otherwise they are *nonuniform*. The fields of physical quantities may show no variation with time, i.e., they may be *stationary*, and may vary with time, i.e., be *nonstationary*.

The field of the scalar quantity φ (e.g., temperature, pressure, density) is specified analytically as a function of the coordinates of field points and time ($\boldsymbol{r}$ is the instantaneous radius-vector of the point M of space)

$$\varphi = \varphi(x, y, z; t) = \varphi(M; t) = \varphi(\boldsymbol{r}; t) \tag{1}$$

Here, time is considered to be a parameter that denotes separate instants. The fields pertain to these when they are nonstationary and can be omitted when the fields are stationary.

The field of the vector quantity $\boldsymbol{a}$ (velocity, acceleration, force, etc.) is prescribed by the vector function of the coordinates of points M and time

$$\boldsymbol{a} = \boldsymbol{a}(x, y, z; t) = \boldsymbol{a}(M; t) = \boldsymbol{a}(\boldsymbol{r}; t) \tag{2}$$

or by three projections of the vector $\boldsymbol{a}$

$$a_x = a_x(x, y, z; t) \qquad a_y = a_y(x, y, z; t) \qquad a_z = a_z(x, y, z; t) \tag{3}$$

To shorten the amount of formulation, letter indexing of coordinates and vector projections is often replaced by *numerical indexing* assuming that

$$x = x_1 \quad y = x_2 \quad z = x_3 \qquad a_x = a_1 \quad a_y = a_2 \quad a_z = a_3$$

and Equation 3 will be replaced by

$$a_i = a_i(x_1, x_2, x_3; t) \qquad (i = 1, 2, 3)$$

Furthermore, proceeding as advised by Einstein, omit the summation sign $\sum$ in front of monomials being summed up if the index, over which the summation is performed, appears twice in the monomial. Thus, for example,

$$c_k = \sum_{l=1}^{3} a_l b_{lk} = a_l b_{lk} \qquad (k = 1, 2, 3)$$

$$f_{pq} = \sum_{l=1}^{3} \sum_{m=1}^{3} a_{pl} b_{qm} c_{lm} = a_{pl} b_{qm} c_{lm} \qquad (p, q = 1, 2, 3)$$

The indices that appear twice and over which summation is carried out are called "dummy indices"; they can be denoted arbitrarily in the process of calculations:

$$c_k = a_l b_{kl} = a_m b_{km} \qquad (k = 1, 2, 3)$$

The rest of the indices (as, for example, k in the previous formula) remain unchanged in each given process of calculation and are called "free indices".

Let us recall the analytical geometry formulas, which will be needed later, having expressed them in the notation just adopted.

Consider two systems of *rectangular* Cartesian coordinates: $0x_1x_2x_3$ and $0x_1'x_2'x_3'$ (Figure 1), which have the common origin at point 0, but which are rotated arbitrarily with respect to one another. We shall denote the *cosines* of the angles between "new" (primed) and "old" axes by α_{kl} according to the matrix

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ x_1' & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ x_2' & \alpha_{21} & \alpha_{22} & \alpha_{23} \\ x_3' & \alpha_{31} & \alpha_{32} & \alpha_{33} \end{array} \tag{I}$$

In each coordinate system we refer to its *set of unit vectors directed along the coordinate axes* as the *basis* of the coordinate system. In the present case there are two such bases: $\boldsymbol{i}_1$, $\boldsymbol{i}_2$, $\boldsymbol{i}_3$ and $\boldsymbol{i}_1'$, $\boldsymbol{i}_2'$, $\boldsymbol{i}_3'$. The radius-vector $\boldsymbol{r}$ of the point M can be expressed in terms of these bases as

$$\boldsymbol{r} = x_1\boldsymbol{i}_1 + x_2\boldsymbol{i}_2 + x_3\boldsymbol{i}_3 = x_1'\boldsymbol{i}_1' + x_2'\boldsymbol{i}_2' + x_3'\boldsymbol{i}_3' \tag{4}$$

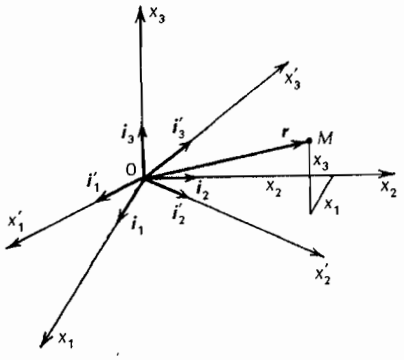

Fig. 1

Form the scalar product of both sides of this equality successively with i_1, i_2, i_3 and i_1', i_2', i_3' by recalling that

$$i_k \cdot i_l = i_k' \cdot i_l' = \delta_{kl} = \begin{cases} 1 & k = l \\ 2 & k \neq l \end{cases} \tag{5}$$

and noting that according to the matrix (I) above:

$$i_k' \cdot i_l = \alpha_{kl} \tag{6}$$

Thus, we shall obtain the following formulas for transforming the "old" coordinates x_1, x_2, x_3 into "new" coordinates x_1', x_2', x_3':

$$x_k' = \alpha_{k1}x_1 + \alpha_{k2}x_2 + \alpha_{k3}x_3 = \alpha_{kl}x_l \quad (k = 1, 2, 3) \tag{7}$$

and vice versa:

$$x_k = \alpha_{1k}x_1' + \alpha_{2k}x_2' + \alpha_{3k}x_3' = \alpha_{lk}x_l' \quad (k = 1, 2, 3) \tag{8}$$

Taking into account that the projections of the vector $\boldsymbol{a}$ on the coordinate axes a_1, a_2, a_3 or a_1', a_2', a_3' are given by the differences between the coordinates that locate the end and beginning of the segment that represents the vector, it can be concluded that the formulas for the vector projections in conversion from one coordinate system to the other will be analogous in form

$$a_k' = \alpha_{kl}a_l \qquad a_k = \alpha_{lk}a_l' \quad (k = 1, 2, 3) \tag{9}$$

Note that Equation 9 is valid only in orthogonal coordinate systems that will be employed below almost without exception.

Let us agree that *physically objective* will be the name given to those quantities which, by virtue of their physical essence, *are independent of the choice of the coordinate system* in which they are represented analytically. Thus, for example, the physical scalar (temperature, pressure, density, etc.) prescribed by its field, i.e., by the function of the coordinates $\varphi(x_1, x_2, x_3)$, should be an *invariant* which satisfies the equality

$$\varphi(x_1, x_2, x_3) = \varphi(x_1', x_2', x_3') = \text{inv}$$

where it is presumed that the sets of "old" and "new" coordinates correspond to the same point of space.

Analogously, a vector prescribed by its three projections $a_k(x_1, x_2, x_3)$ $(k = 1, 2, 3)$, or, in another coordinate system, by the projections $a'_k(x'_1, x'_2, x'_3)$ $(k = 1, 2, 3)$ should be independent of the coordinate axes for it to be capable of representing a physical quantity.

In order to find the conditions for the physical objectivity of the vector function $\boldsymbol{a}$, we shall first prove that the scalar product $\boldsymbol{a} \cdot \boldsymbol{b} = a_k b_k$, where a_k and b_k satisfy the conversion formulas of Equation 9, is independent of the directions chosen for the coordinate axes, in which these multiplies were determined (i.e., where it is an *invariant*).

According to Equation 9, we have (k is a dummy index)

$$a_k b_k = \alpha_{lk} a'_l \alpha_{mk} b'_m = \alpha_{lk} \alpha_{mk} a'_l b'_m \tag{10}$$

To conform with the well-known theorems of analytical geometry, the cofactor $\alpha_{lk}\alpha_{mk}$ on the right side of Equation 10 is the expression of the cosine of the angle between the axes $0x'_l$ and $0x'_m$ in terms of the cosines of the angles between these axes and those of the system $0x_1x_2x_3$. In compliance with the orthogonality property of the coordinate axes

$$\cos(\widehat{x'_l, x'_m}) = \alpha_{lk}\alpha_{mk} = \begin{cases} 1 & \text{when} \quad m = l \\ 0 & \text{when} \quad m \neq l \end{cases} \tag{11}$$

Analogously, expressing the cosine of the angle between the axes $0x_l$ and $0x_m$ in terms of the cosines of the angles about the axes of the system $0x'_1x'_2x'_3$, we shall obtain

$$\cos(\widehat{x_l, x_m}) = \alpha_{kl}\alpha_{km} = \begin{cases} 1 & \text{when} \quad m = l \\ 0 & \text{when} \quad m \neq l \end{cases} \tag{12}$$

Taking into account Equation 11, we shall obtain the equality that proves the *invariance of the scalar product* $\boldsymbol{a} \cdot \boldsymbol{b}$

$$a'_k b'_k = a_l b_l = \text{inv} \tag{13}$$

Assuming that in this equality $\boldsymbol{b} = \boldsymbol{a}$, we may note the *invariance of the sum of the squared projections*

$$a'_k a'_k = a_l a_l = \text{inv} \tag{14}$$

i.e., squared length (absolute value) of the vector and, consequently, of the very length (absolute value) of the vector. Bearing in mind that

$$a_k b_k = \boldsymbol{a} \cdot \boldsymbol{b} = ab \cos(\widehat{\boldsymbol{a}, \boldsymbol{b}})$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are invariants, and specifying different vectors $\boldsymbol{b}$, we shall also note the invariance of the location of the vector $\boldsymbol{a}$ relative to the points of space.

The role of Equation 9 as conditions for the physical objectivity in the analytical determination of the vector is clear. As regards the formulas of conversion from one coordinate system to the other (Equation 7 and 8), they constitute the specific case of Equation 9 in application to the radius-vector $\boldsymbol{r}$ of point M and, consequently, they establish the physical objectivity, in analytical determination of the radius-vector of the point, with the aid of its coordinates.

Calling attention to the identity of the formulas of conversion from one coordinate system to the other for the Cartesian coordinates themselves (Equations 7 and 8) and for vector projections (Equation 9), we can formulate the conditions for the physical objectivity in the analytical representation of vectors: In converting from one rectangular Cartesian coordinate system to the other, the projections of a physical vector must be transformed in the same way as the coordinates.

2. A Second-Rank Tensor: Conditions for the Physical Objectivity of Its Analytical Specification – A Dyad and a Rotation Tensor

Along with scalar quantities, the field of which is determined by one numerical function, and vector quantities with the field specified by three numerical functions (vector projections onto the coordinate axes), the mechanics of continua also employ more mathematically complex physical quantities, i.e., *second-rank tensors.*[1]

The second-rank tensor can be defined by two independent techniques leading to identical results.

Generalizing the notions of the scalar and vector and the conditions for their physical objectivity, we shall determine the second-rank tensor T at each point of a three-dimensional space as a set of nine quantities T_{pq} $(p, q = 1, 2, 3)$ which are the tensor components specified in a certain rectangular Cartesian coordinate system and which, when converting to another coordinate system, are transformed by the formulas (r and s are dummy indices)

$$T'_{pq} = \alpha_{pr}\alpha_{qs}T_{rs} \qquad T_{pq} = \alpha_{rp}\alpha_{sq}T'_{rs} \quad (p, q = 1, 2, 3) \tag{15}$$

These equalities can be interpreted as the formulae of transition from one rectangular Cartesian coordinate system to the other from the product of two coordinates. In fact,

$$x'_p x'_q = \alpha_{pr}x_r\alpha_{qs}x_s = \alpha_{pr}\alpha_{qs}x_r x_s$$

$$x_p x_q = \alpha_{rp}x'_r\alpha_{sq}x'_s = \alpha_{rp}\alpha_{sq}x'_r x'_s$$

in agreement with Equations 15.

Thus, according to the first definition, the second-rank tensor represents the set of nine quantities (tensor components) which are transformed as a *product of two coordinates* when converting from one coordinate system to the other. The second definition of the tensor is based on the concepts of *linear vector-function* or *linear transformation of the vector.* Both are expressed analytically as the equalities (l is a dummy index, k is a free index)

$$b_k = T_{kl}a_l \quad (k = 1, 2, 3) \tag{16}$$

where a_i and b_i are the coordinates of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$; T_{kl} are the coefficients of the linear vector-function or of linear transformation.

The second definition of the second-rank tensor states: if in the linear transformation (linear vector-function; Equation 16) to the physical vector $\boldsymbol{a}$, the physical vector $\boldsymbol{b}$ also corresponds, then the set of the transformation coefficients T_{kl} $(k, l = 1, 2, 3)$ represents a physically objective quantity, i.e., the second-rank tensor.

[1] Sometimes "order" rather than "rank" is used in this Context, i.e., "second-order tensor".

$$\Gamma_{BC} = -\left(a_1 + \frac{\partial a_1}{\partial x_2}\delta x_2\right)|BC| = -\left(a_1 + \frac{\partial a_1}{\partial x_2}\delta x_2\right)2\delta x_1$$

$$\Gamma_{CD} = -\left(a_2 - \frac{\partial a_2}{\partial x_1}\delta x_1\right)|CD| = -\left(a_2 - \frac{\partial a_2}{\partial x_1}\delta x_1\right)2\delta x_2$$

$$\Gamma_{DA} = \left(a_1 - \frac{\partial a_1}{\partial x_2}\delta x_2\right)|DA| = \left(a_1 - \frac{\partial a_1}{\partial x_2}\delta x_2\right)2\delta x_1$$

The summation will yield

$$\Gamma_{ABCD} = \left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right)4\delta x_1 \delta x_2 = \mathrm{rot}_3 \boldsymbol{a}\, \delta\sigma \tag{117}$$

where $\delta\sigma$ denotes the area of the rectangle $ABCD$ $\delta\sigma = 2\delta x_1 \cdot 2\delta x_2 = 4\delta x_1 \delta x_2$. Equality (117) shows that the circulation of the vector $\boldsymbol{a}$ along the contour of the rectangle is equal to the flux of the vector rot $\boldsymbol{a}$ through the rectangle area. This formula can be applied to any element $\delta\sigma$ of the surface σ in the form

$$\delta\Gamma = \mathrm{rot}_n \boldsymbol{a}\, \delta\sigma \tag{118}$$

Summing up both sides of Equation 118 over all the elements of the surface σ, note that circulations along the common sides of adjacent cells cancel out and only circulations along those sides of the cells remain that constitute the contour C. We shall obtain in the limit

$$\sum \delta\Gamma = \oint_C \delta\Gamma = \oint_C \boldsymbol{a} \cdot \delta\boldsymbol{r}$$

The sum of the right sides in Equation 118 will be reduced to the *total flux of the vector* rot $\boldsymbol{a}$ *through the surface* σ. This proves the validity of Stokes, theorem.

The basic concepts and formulas of vector and tensor analysis that have been given in this chapter will suffice for understanding standing the subject matter of the book as a whole. For a more extensive account of the foundations of vector and tensor calculus, reference may be made to the following special textbooks:

1. Kochin, N. Ye., *Vektornoye ischisleniye i nachala tenzornogo ischisleniya (Vector Calculus and the Laws of Tensor Calculus)*, Nauka Press, Moscow, 1965.
2. Sedov, L. I., *Mekhanika sploshnoi sredy (Mechanics of a Continuous Medium)*, Part 1, Nauka Press, Moscow, 1983; *A Course in Continuum Mechanics*, Vol. 1 to 4, Wolters-Nordhoff Publishing, Amsterdam, 1971 to 1972.
3. Luriye, A. I., *Teoriya uprugosti (The Elasticity Theory)*, Nauka Press, Moscow, 1970 (Appendices I, II, III, pp. 799 to 870).
4. Mase, G. E., *Teoriya i zadachi mekhaniki sploshnykh sred (The Theory and Problems of the Mechanics of Continua)*, Mir Press, Moscow, 1974.

The previous (5th) edition (Nauka Press, Moscow, 1978) of the present book contains the summary of the most popular formulas in the vector and tensor calculi.