HYDRODYNAMICS: EXAMPLES AND PROBLEMS

A Textbook

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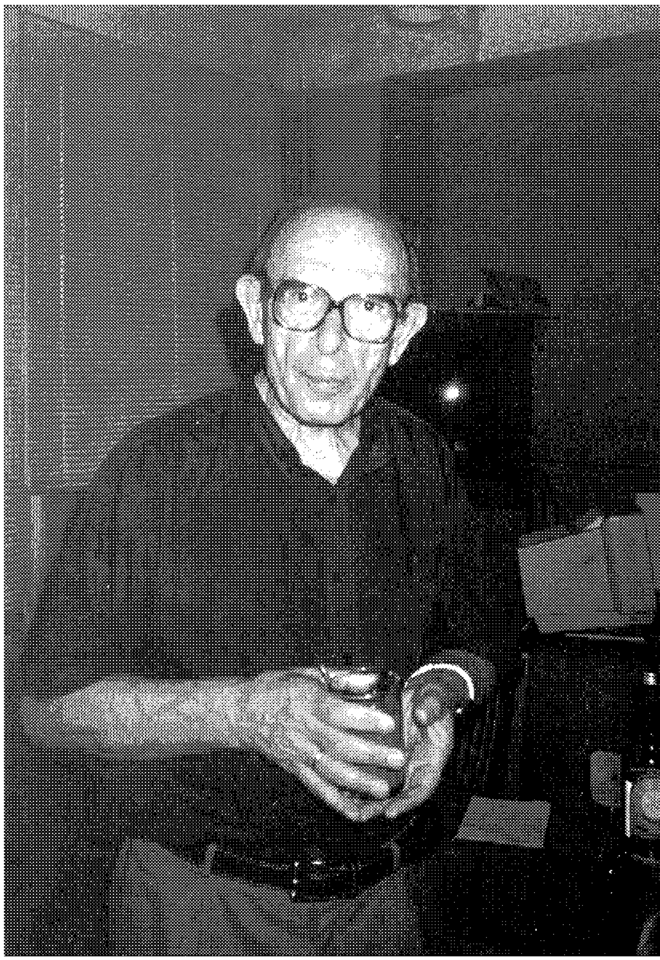
To the memory of Professor Doctor Yuri A. Buyevich who always kept love for life and science.



May 1997. At the Department of Mathematical Physics,
Urals State University, Ekaterinburg.

Prof. Dr. Yuri Buyevich died after serious illness on December 22, 1998. An outstanding scientist in the field of applied mechanics of continua, physical-chemical hydrodynamics, and heat–mass exchange, he was a man of great erudition, vast scientific activity, and enormous capacity for work. Yuri Buyevich also had a reputation as a prolific generator of ideas. It was a pleasure to work with him.

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PREFACE

This book contains examples and problems in hydrodynamics that were an integral part of a theoretical course delivered at the Ural State University over many years. There are more than 200 examples and problems supplemented by answers and solutions.

In this textbook we primarily propose the problems, the physical content of which is rather transparent, and the process of solving allows the reader to see all the beauty of hydrodynamics. We have deliberately excluded problems whose solution might require rigorous mathematical proofs of different theorems and statements.

The first chapter is devoted to the foundations of tensor calculus. Although at first the reader may wish to skip over this chapter, we included it because in studies of hydrodynamics one uses existing equations written, as a rule, in Cartesian, spherical, and cylindrical coordinates. And here, two difficulties emerge: first, the need for an understanding of how these equations are derived, and second, facility with writing them for other symmetries (parabolic, elliptical, etc.), since a wide variety of symmetries are encountered in studies of natural phenomena, and it is the most convent to use the appropriate curvilinear coordinates. With the answers to problems from the first chapter, readers can write the Navier–Stokes equations and the continuity equation in all coordinates considered in the text; moreover, if they understand how these equations are derived, it will be not a particular problem to write the equations in question in any other curvilinear coordinates.

The second and third chapters are devoted to dimensional analysis and self-similar solutions. We deliberately placed these chapters before consideration of the basic issues of hydrodynamics: inviscid and viscous fluids. The reasons are the following. First, the methods outlined in the second and third chapters often allow one to solve intricate problems easily. Second, dimensional analysis and self-similar solutions are of interest in themselves, and appear helpful in different branches of modern physics. Several examples are proposed. At this point we thank professors V. V. Mansurov and V. S. Nustrov for help in writing the first three chapters.

The fourth and fifth chapters, which discuss inviscid and viscous fluids, numerous examples that demonstrate workable approaches to analytically solvable problems.

During our work on this book we used many original sources. Some of them are rare or may be inaccessible to the reader, others are accessible only to Russian readers.

We hope that this book will help students to understand and master the main issues of hydrodynamics. Also we hope that it will be useful for lecturers and instructors.

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NOMENCLATURE

$\mathbf{f}$	density of mass forces
g	acceleration of gravity
i	imaginary unit
J_ν	Bessel functions of the first kind
k	wavenumber
P_l^m	Legendre polynomials
p	pressure
s	Laplace variable
t	time
$\mathbf{v}$	fluid velocity
Y_ν	Bessel functions of the second kind
x, y, z	Cartesian coordinates

Greek Symbols

$\Gamma(q)$	Euler's gamma function
$\Gamma^j{}_{ik}$	Christoffel symbols of the second kind
Δ	Laplacian
δ^{ik}	Kronecker delta
δ^*	displacement layer thickness
δ^{**}	loss of momentum thickness
η	dynamic viscosity
λ	wavelength
ν	kinematic viscosity
ρ	fluid density
Φ	velocity potential
Ψ	stream function

$\mathbf{\Omega}$	rotor of velocity $\mathbf{v}$
ω	frequency

Subscript

∞	the value at infinity

Superscripts

$*$	Laplace transform
$-$	average value or complex conjugation
r	radial component
x	x component
y	y component
z	z component
θ, φ	transverse and polar component, respectively

CHAPTER

ONE

TENSOR ANALYSIS

1.1 CONTRAVARIANT AND COVARIANT COMPONENTS OF TENSORS

Continuous manifolds met in geometry and mechanics determine the properties of geometrical objects as well as the positions and motion of points of material systems. These manifolds are called many-dimensional spaces. In this book we use only spaces of three or fewer dimensions. In analytical studies of some simple properties of tensor values, we will use oblique rectilinear coordinates. A coordinate basis will be defined by means of a system of noncoplanar vectors $\mathbf{e}_i$ $(i = 1, 2, 3)$. It is significant that moduli of these vectors may differ from unity. The condition of noncoplanarity is equivalent to the condition of linear independence of vectors $\mathbf{e}_i$. Let us define the scalar (inner) product of basis vectors $\mathbf{e}_i$ with the aid of coefficients of a symmetric positive defined quadratic form

$$g_{ik}\Delta x^i \Delta x^k$$

namely,

$$(\mathbf{e}_i, \mathbf{e}_k) = g_{ik}$$

where parentheses denote the scalar (inner) product. It can be shown that coefficients g_{ik} define a space metric. We will denote the coordinates of an arbitrary point M of the space by x^i $(i = 1, 2, 3)$. The set of coordinates x^1, x^2, x^3 uniquely defines the point $M(x^i)$. Any vector $\mathbf{a}$ is given by

$$\mathbf{a} = \mathbf{e}_1 a^1 + \mathbf{e}_2 a^2 + \mathbf{e}_3 a^3 = \sum_{i=1}^{3} \mathbf{e}_i a^i$$

For the sake of simplicity, the following convention will be used. If summation is performed over one or more pairs of upper and lower indices, the summation symbol is omitted. That is,

$$\sum_{i=1}^{3} a^i b_i = a^i b_i$$

Then the preceding expression for the vector $\mathbf{a}$ may be rewritten as follows:

$$\mathbf{a} = \mathbf{e}_i a^i$$

This designation is extended to double and more complex sums, for example,

$$\sum_{i=1}^{3} \sum_{k=1}^{3} A_{ik} a^i b^k = A_{ik} a^i b^k$$

If summation is performed over a pair of indices, these indices are called umbral. The letter notation of umbral indices can be changed in an arbitrary way. It is important to keep in mind that umbral and significant indices should not be mixed.

The values a^i corresponding to the expansion of a vector $\mathbf{a}$ in an oblique coordinate system are called contravariant components of this vector $\mathbf{a}$ (the purpose of introducing of this term will be made clear below). In addition, any vector $\mathbf{a}$ can also be determined by the system of values a_1, a_2, a_3, called covariant components of the vector. These values are components in the expansion in terms of a basis $\mathbf{e}^k$ (which is reciprocal to the basis $\mathbf{e}_i$) defined as follows:

$$\mathbf{e}^k = g^{ki} \mathbf{e}_i$$

where g^{ki} are elements of the matrix reciprocal to g_{ik}; that is,

$$g^{ik} = \frac{G^{ik}}{g} = \frac{1}{g} \frac{\partial g}{\partial g_{ik}}$$

Here G^{ik} is a cofactor of g_{ik} and $g = \det\{g_{ik}\}$. With regard to what has been said, the scalar product of vectors $\mathbf{a}$ and $\mathbf{b}$ can be presented in the form

$$(\mathbf{a}, \mathbf{b}) = (\mathbf{e}_i, \mathbf{e}_k)a^i b^k = g_{ik}a^i b^k = g^{ik}a_i b_k = a^i b_i$$

1.2 TRANSFORMATION OF COMPONENTS OF A VECTOR

Let us consider relations between components of a vector in two different coordinate systems defined by an old $\mathbf{e}_i$ and a new $\mathbf{e}_k'$ bases. We will define a transformation of the two coordinate systems by formulas of direct and inverse transformations of base vectors

$$\mathbf{e}_j' = \alpha_j^{\ i}\mathbf{e}_i$$

$$\mathbf{e}_i = \beta_i^j \mathbf{e}_j'$$

Here $\alpha_j^{\ i}$ and β_i^j are coefficients of the direct and inverse transformation, respectively. It can be shown that there exist the following relations between contravariant and covariant components of a vector $\mathbf{a}$ in both old and new bases

$$a'^j = \beta_i^j a^i$$

$$a^i = \alpha_j^{\ i} a'^j$$

$$a_j' = \alpha_j^{\ i} a_i$$

$$a_i = \beta_i^j a_j'$$

where primes correspond to the components in the new basis. Thus a direct transformation of contravariant components of a vector is made by coefficients of an inverse transformation of base vectors, whereas an inverse transformation of contravariant components of a vector is made by coefficients of a direct transformation of base vectors. This explains the origin of the term "contravariant components".

Analogously, a direct transformation of covariant components of a vector is made by coefficients of a direct transformation of base vectors, whereas an inverse transformation of covariant components of a vector is made by coefficients of an inverse transformation of base vectors. This explains the origin of the term "covariant components".

1.2.1 Definition of a Tensor

Any physical or geometrical quantity defined by a set of values or functions that are transformed under change of the coordinate system as follows:

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