GRAPHS of ELEMENTARY and SPECIAL FUNCTIONS

HANDBOOK

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Preface

The notion of a functional dependence is one of the most important notions in modern mathematics. In studying processes in nature and solving engineering problems, one encounters a situation in which one quantity varies depending on another quantity.

Among the many ways to define the function (analytical, tabular, descriptive), defining it by its graph is widely used because of its visual presentation. In some cases, this method of defining the function is the only one possible. However, it is surprising that so far there are not many reference books that describe different aspects of studying and plotting graphs of various functions, in particular, with the use of methods of calculus.

This reference book gives essentials on functions and methods of plotting their graphs. Special attention is paid to methods used for plotting graphs of functions that are defined implicitly and in a parametric form, methods for making plots in polar coordinates, and plotting graphs of remarkable curves, special functions, etc.

The book consists of two parts. The first part gives general information about the notion of a function and methods of plotting graphs of functions without the use of derivatives. The second part deals with methods of studying functions and plotting their graphs with the use of calculus. The book contains many examples of functions that are more complicated than commonly considered, graphs of important curves, and properties, and graphs of common special functions (gamma function, integral exponential functions, Fresnel integrals, Bessel functions, orthogonal polynomials, elliptic functions, Mathieu functions, etc.).

The book contains 760 figures and uses material found in existing references, textbooks, articles, and other sources on the subject.

Part I

Plots of graphs with elementary methods

Chapter 1

Basic notions of numbers, variables, and functions

1.1. Numbers. Variables. Functions

In this book we will occasionally use certain symbols to denote logical expressions. Symbol (generality quantifier) will replace the expressions "for arbitrary," "for any," etc., symbol $\exists$ (existential quantifier) — the expressions "there exists," "one can find," and so on.

The notation $A \Rightarrow B$ (implication symbol) means that statement A implies statement B.

The notation $A \Leftrightarrow B$ (equivalence symbol) means that $A \Leftarrow B$ and $B \Rightarrow A$.

The notation $A \wedge B$ (conjunction quantifier) means that both statements A and B hold.

If one of the statements A or B holds, then we use notations $A \vee B$ (disjunction quantifier).

1.1.1. Real numbers

All infinite decimal fractions (positive, negative, and zero) make up the set of *real numbers* R (continuum of numbers). This set can be subdivided into two parts (subsets): the subset of rational numbers (integers, fractions, both positive and negative, and the number 0) and the set of irrational numbers. Every rational number can be written as a fraction $\pm p/q$, where p and q are natural numbers, i.e., 1, 2, ..., n, ... A rational number is a periodic decimal fraction.

An irrational number is a nonperiodic infinite decimal fraction. For example, 1/3 = 0.333 ..., 5 = 5/1 are rational numbers; $\pi = 3.14159...$, $\sqrt{2} = 1.4142...$ are irrational.

Basic properties of the set of real numbers

The set of real numbers is *ordered*, i.e any two numbers from this set a, b are either equal or one is greater than the other. The order relation is denoted by $a < b$ (a is less than b), $a \le b$ (a is not greater than b), $a > b$ (a is greater than b), $a \ge b$ (a is not less than b).

The set of real numbers is *dense*, i.e. for any two arbitrarily close real numbers a and b there are infinitely many real numbers (both rational and irrational) lying between these numbers.

The set of real numbers is *continuous*, i.e. any Dedekind cut defines a unique real number a which separates the numbers belonging to the class A from the numbers belonging to the class

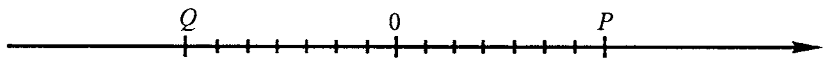

Figure 1.1. Number axis. O is the origin; points P and Q lie on the axis.

B, that is, a is the greatest number in the class A (and then the class B does not contain the smallest number), or it is the smallest number in the class B (then class A does not have the greatest number).

A *Dedekind cut* D of the set of real numbers is a splitting of all real numbers into two classes, the lower class A and the upper class B, in such a way that each real number belongs to only one class and each number from class A is less than an arbitrary number from class B.

It is convenient to draw real numbers as points on the real axis. The *real axis* is a line with an initial point O, direction, which is taken to be positive, and a scale (a line segment the length of which is taken as one). To every real number there corresponds a unique point on the real axis. The number 0 corresponds to the point O. If a number x is positive, then the corresponding point P lies to the right of the point O with the distance OP equal to x. If a number x is negative, then it corresponds to a point Q which lies to the left of O with the distance OQ equal to $-x$ (Figure 1.1). The converse statement is also true: every point in the real axis corresponds to a unique real number. This means that there is a one-to-one correspondence between real numbers and points on the real axis. This allows us in many instances to use interchangeably the notions "a number x" and "a point x." The following statement will be of practical use: every irrational number can be arbitrarily well approximated by rational numbers.

Real numbers can be subdivided into *real algebraic numbers* (real roots of algebraic equations with integer coefficients) and *real transcendental numbers* (the rest of the real numbers).

Any finite or infinite collection of real numbers is called a *number set*. Sets will be denoted by capital letters $X, Y, Z, \ldots$, and we use small letters $x, y, z, \ldots$ to denote numbers that belong to a set. For example $x \in X$ (x belongs to X). If an element x does not belong to a set X, we write this as $x \notin X$. The notation $X = \{x\text{:condition } K \text{ holds}\}$ is used to denote the set of all elements which verify condition K.

Let us consider two sets A and B. If each element of the set A is an element of the set B, then we say that A is a *subset* of the set B and denote this by $A \subset B$ (or $B \supset A$). If $A \subset B \wedge B \subset A$, then the sets A and B are called equal. This fact is denoted by $A = B$. The set which does not contain any element is called the *empty set* and is denoted by $\varnothing$. Any subset contains the empty set as a subset.

The simplest operations one can perform with sets are taking unions (sums), intersections (products), and differences.

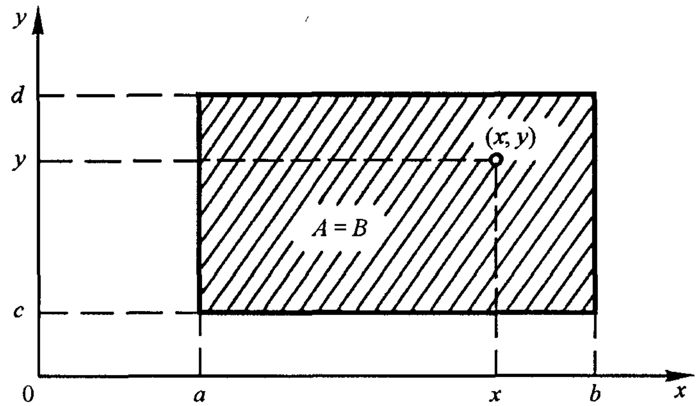

Figure 1.2. Direct product of sets A and B.

By the *union* (or sum) of sets A and B, we call the set $C = A \cup B = \{x : x \in A \vee x \in B\}$.

The *intersection* (or product) of sets A and B is the set $D = A \cap B = \{x : x \in A \wedge x \in B\}$.

The *difference* of sets A and B is the set $F = A \setminus B = \{x : x \in A \wedge x \notin B\}$.

The *symmetric difference* of two sets A and B is the set $L = A \Delta B = (A \cup B) \setminus (A \cap B)$.

A pair of elements a and b is called the *ordered pair* and is denoted by (a,b) if the order in which they appear is given. By this definition, $(a,b) = (c,d) \Leftrightarrow (a = c) \wedge (b = d)$.

Let $A \subset M$, $B \subset M$. The totality of all possible ordered pairs (a,b), where $a \in A$, $b \in B$, is called the *direct product* of the sets A and B, and is denoted by $A \times B$. For example, let $A = \{x : a \le x \le b\}$, $B = \{y : c \le y \le d\}$. Then the direct product $A \times B$ is in one-to-one correspondence with the set of points (x, y) that make the rectangle (see Figure 1.2).

Among number sets, the following sets will be of importance.

A bounded closed interval (line segment) $X = [a;b]$. The set X contains all the numbers x which satisfy the inequality $a \le x \le b$. On the real axis this set forms the line segment ab, with the endpoints included (Figure 1.3 a).

A bounded open interval (line segment) $X = (a;b)$. The set X contains all numbers x which satisfy the inequality $a < x < b$. On the real axis, this set is the line segment ab without the endpoints (Figure 1.3 b).

Bounded half-open (or half-closed) intervals $[a;b)$ and $(a;b]$. These are sets X which contain numbers x for which $a \le x < b$ and, respectively, $a < x \le b$. The corresponding interval on the real axis contains the left endpoint and does not contain the right endpoint, and, respectively, does not contain the left endpoint and contains the right endpoint.

One-sided infinite intervals $(b; \infty)$, $[b; \infty)$, $-\infty; a)$, $(-\infty; a]$. These are sets X of numbers x for which $x > b, x \ge b, x < a, x \le a$ respectively. On the real line these sets correspond to respective infinite half-axes.

Figure 1.3. (*a*) segment $[a;b]$, (*b*) interval $(a;b)$.

The set of all real numbers (the real axis) is also considered as an interval and is denoted by $(-\infty; +\infty)$.

In the sequel, we will be using the following notations. The set of all natural numbers will be denoted by **N**, the set of all nonnegative integers — $\mathbf{Z}_0$, the set of all integers — **Z**, the set of all rational numbers — **Q**, the set of all real numbers — **R**, the set of all complex numbers — **C**.

Let a be an arbitrary number. A δ-*neighborhood* of the point $x = a$ is the open interval $(a-\delta;a+\delta)$ containing the point a. In other words, a δ-neighborhood of a point a is the set of all x for which $|x-a| < \delta$, where δ is a positive number (Figure 1.4). A *neighborhood* of a point $x = a$ is an arbitrary set containing a δ-neighborhood of this point for some $\delta > 0$.

The *absolute value* of a real number a, denoted by $|a|$, is the nonnegative number defined by:

$$|a| = \begin{cases} a, & \text{if } a > 0, \\ -a, & \text{if } a < 0, \\ 0, & \text{if } a = 0. \end{cases}$$

The main properties of the absolute value are the following:

$$|a| \geq 0,$$
$$|a| = 0 \quad \text{implies} \quad a = 0,$$
$$\big||a| - |b|\big| \leq |a+b| \leq |a| + |b|,$$
$$\big||a| - |b|\big| \leq |a-b| \leq |a| + |b|,$$
$$|ab| = |a|\,|b|, \qquad \left|\frac{a}{b}\right| = \frac{|a|}{|b|} \quad (b \neq 0),$$
$$\text{if} \quad |a| \leq A \quad \text{and} \quad |b| \leq B, \quad \text{then} \quad |a+b| \leq A+B, \quad |ab| \leq AB.$$

Consider a nonempty set X. An arbitrary number M such that $x \leq M$ for all $x \in X$ is called the *upper bound* of the set X. The *lower bound* of a set X is a number m such that $x \geq m$ for all $x \in X$.

Sets that have an upper bound are called *bounded from above*. If a set has a lower bound, then it is called *bounded from below*. A set which is bounded from below and above is called *bounded*.

The smallest among the numbers M is called the *least upper bound*, and the largest among the numbers m is called the *greatest lower bound*. More precisely, a number

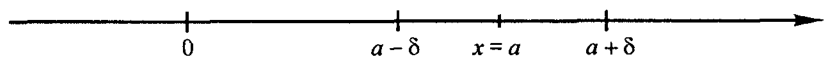

Figure 1.4. δ-neighborhood of point *a*.

$M^* = \sup X = \sup\{x\}$ is called the least upper bound of a set $X = \{x\}$ if for all $x \in X, x \le M^*$, and all $\varepsilon > 0$ there exists $x_0 \in X$ such that $M^* - \varepsilon < x_0 \le M^*$. The number $m^* = \inf X = \inf\{x\}$ is called the greatest lower bound of a set $X = \{x\}$ if for all $x \in X$, we have $x \ge m^*$, and for every $\varepsilon > 0$ there exists $x \in X$ such that $m^* \le x < m^{*+}\varepsilon$.

If a set X is bounded from above (below), then it has the least upper (greatest lower) bound. This means that if a set is bounded, then it has the least upper bound and the greatest lower bound.

1.1.2. Constants and variables

While observing and studying processes in nature, one sees that some quantities vary, i.e. their numeric values change, whereas the others remain constant. For example, in a uniform motion, the time and distance change, but the velocity remains constant. If a gas is heated in a closed vessel, then the gas pressure and temperature change, but the mass and the volume of the gas are unchanged.

A *variable* is a quantity that can assume different values in a problem under consideration. A quantity that does not change in a problem under consideration is called *constant.* Quantities that remain constant in all problems are called *absolute constants.* An example of an absolute constant is $\pi = 3.141...$.

Variables can change in various ways. Some variables can take only integer positive values, others — infinite, negative values, etc. One also considers *discrete* variables, — those which take values from a finite or infinite set of isolated values, and *continuous* variables that, assuming two values $x = a$ and $x = b$, take also all the values x with $a < x < b$.

The set of numbers from which a variable x takes its values is called the *value domain* of the variable x. A variable is called *increasing* if each of its next values is greater than the previous one. A variable is called *decreasing* if each of its next values is less than the previous one. Increasing and decreasing variables are called *monotone.* A variable is *bounded* if its value domain is a bounded set.

1.1.3. Concept of a function

Often we have to consider not variables by themselves but a relationship between them, a dependence of one variable on another. Actually there are no variables in nature that would

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