# PREFACE

The fundamentals of the theory of ill-posed problems, the regularization theory, were developed in the 1950's – 1960's by Soviet mathematicians A. N. Tikhonov, V. K. Ivanov and M. M. Lavrentiev. Its rapid development in recent years, especially, after publication of A. N. Tikhonov's fundamental articles [190, 191] can be attributed, first, to continuous expansion of the field of practical applications of the theory and second, to substantial advances in computer science. The wide use of computational experiment which reduces not only the time but also the cost of research and development has encouraged practical application of the ill-posed problem theory. The computational experiment is of special importance in computerization of research and design of structures in modern engineering. The problems of structural and parametric identification of mathematical models often appear ill-posed. Ill-posed problems frequently arise in data processing in physical experiments.

In recent years a number of theoretical monographs concerned with the theory of ill-posed problems have been published [72, 90, 117, 122, 142, 187, 193, 196]. However, the monographs actually ignore one of the promising direction in the development of the regularization theory which is widely used, in particular, when solving thermophysical problem. That direction, which can be called "iterative regularization", consists in the construction of regularizing algorithms on the basis of various iterative methods, with an iteration number as the regularization parameter. Many iterative methods, including gradient techniques, are rather resistant to errors in the input data: in the initial iterations the produced approximations differ very little from the corresponding approximations obtained with exact input data and the errors gradually increase as the iteration number rises. Therefore, it is natural to try to get stable approximations, by stopping the iterative process at a certain iteration number consistent with the error in the initial data. M. M. Lavrentiev was the first who used the idea [116].

The simple iteration method is the most convenient for the analysis because it is linear. With exact initial data, its convergence for equations with a continuous linear operator, having no bounded inverse, has been proved by V. M. Friedman [208] who also proved convergence for nonlinear gradient methods of the steepest descent type [207]. B. A. Morozov [140] and A. B. Bakushinsky [40] have obtained complete enough results on the regularizing properties of the simple iteration method. This method was then analyzed in [42, 111, 171, 182, 183, 234, 235]. However, no specific ways were indicated in the publications for the choice of the regularization parameter (the iteration number). The residual criterion for choosing the regularization parameter when the operator is exactly prescribed has been justified in [40, 83]. A thechnique for the choice of the parameter, taking into consideration computational errors, has been suggested and justified in [84]. In [56, 57], techniques for choosing the regularization parameter are considered by accounting errors in both the right-hand side and the operator, in particular, a residual criterion and a generalized residual criterion, estimation of errors in corresponding regularization algorithms is given and their optimality in order is shown. In [61] choice of the regularization parameter based on the iterative process "stacking" is justified when both errors in the operator and the right-hand side and the computational errors are taken into account. It should be noted that error estimates for regularizing algorithms were obtained under the assumption of source-like representability of a desired solution that could be a strong enough condition. For example, in boundary inverse heat conduction problems of reconstructing the boundary condition of the first or second kind $u(\tau)$ defined at the interval $[0, \tau_m]$, the equality $u(\tau_m) = 0$ is the necessary condition for the source- like representability of $u(\tau)$, i.e. at least $u(\tau_m)$ must be known which is extremely rarely occure in practice.

It is known that the gradient minimization methods of the steepest descent and conjugate gradient types are more effective when solving well-posed problems compared with the simple iteration method. However, investigation of applicability of the methods to ill-posed problems is very difficult because of their nonlinearity even for linear problems. Therefore gradient-based regularizing iterative algorithms for linear operator equations have been obtained only recently. Convergence of the gradient methods with exact initial data is analyzed in [68, 207, 228, 231]. Regularizing algorithms built on their basis when errors in the operator and the right-hand side take place, including the justification of the residual criterion and generalized residual criterion, are considered in [24, 25, 69, 70, 169, 174, 175]. Publications [38, 39, 170] deal with taking into account of apriori information on the unknown solution in the regularizing gradient algorithms. A regularizing iterative algorithm besed on the implicit iterative scheme is considered in [198].

It should be emphasized that the methods with a higher convergence rate, such as the Newton methods, are invalid for iterative regularization since in a linear case they are reduced to direct inversion of the operator in the initial equation, but in nonlinear problems they require inversion of the derivative of the operator that has no bounded inverse in ill-posed problems. Of course, this fact does not exclude application of the Newton methods in other regularization forms, e.g., in that analyzed in [41, 59] where regularization is performed by introducing appropriate additives into the iterative algorithm itself.

The above studies concerned with iterative regularization were fundamental for this branch of the theory of ill-posed problems. Consistent presentation of corresponding results and their further development is the main goal of this book.

Whereas the iterative regularization of linear ill-posed problems has been studied at present fully enough, there are actually no publications devoted to nonlinear problems; isolated results are presented in [92, 149]. Meanwhile, as it has been shown by computational experiments, the iterative algorithms for nonlinear ill-posed problems formally based on the same scheme as for the linear problems, appear to be quite effective. The results encourage further investigations of iterative algorithms for nonlinear inverse problems in a general case.

Great attention is paid in the book to the various computational aspects of iterative regularization implementation, particularly, that associated with determination of residual functional gradients, development of modified gradient algorithms for multiparametric problems, including those taking into consideration the qualitative and quantitative apriori information on the unknown quantities.

The second goal of the book is application of the method developed to some ill-posed thermal problems, namely, inverse heat transfer problems. Among them, boundary and coefficient inverse heat conduction problems (IHCP) can be mentioned primarily which certainly do not exhaust the sphere of applications of iterative regularization. IHCPs have been chosen by two reasons. First, it is an important enough class of the problems of mathematical physics that has recently become widely used in some fields of science and technology such as mechanical engineering, aerospace industry, power engineering, metallurgy, etc. Second, inverse heat conduction problems are diverse in forms and statements, ill-posedness degree, and, therefore, they appear to be very convenient for testing the efficiency of methods and algorithms of the regularization theory.

The methodology based on solving inverse problems is a new lead in the investigation of heat and mass transfer processes, development and optimization of thermal conditions of engineering objects and production processes. Rather high interest to solution of these problems is induced by practical needs of including of nonstationary and nonlinear effects in heat and mass transfer processes. These effects restrict essentially the application of classical methods and necessitate the development of new approaches, among which there are methods based on solving inverse heat and mass transfer problems. Their main advantage is that they allow experiments to be conducted in conditions maximally close to real ones, or directly during operation of real objects. Besides, the new approach increases the informativeness, saves experimentation time compared with conventional methods. A reduction of the cost is an important quality of the methods as well.

At present, to solve inverse heat transfer problems algorithms are extensively developed on the basis of the variational technique for construction of regularizing operators, step regularization of approximate analytical and difference forms of solution and step regularization of linear filtration algorithms [2, 3, 5, 9, 51, 85, 105, 106, 112, 130, 131, 147, 151, 179, 195, 211, 214, 221]. Now there is a large amount of relevant publications. They are surveyed in [9, 106, 131]. The analysis of results of computational experiments and practical applications of the above approaches to the solution of inverse heat transfer problems in comparison with iterative regularization has allowed a conclusion to be made that regularizing gradient algorithms are extremely advantageous for inverse problems of various forms and statements, including nonlinear, multidimensional, overdefined ones, because of their simplicity and versatility of algorithmic constructions. Their important strength is the ability to use this approach for problems, whose statements include limitations on the class of admissible solutions. This factor is of special importance since the quality of approximations to the desired solution of an ill-posed problem depends essentially on the completeness of taking into consideration apriori information on that solution.

The accuracy of recovering characteristics from the inverse problem solution can essentially depend on the experiment design, in particular, measurement design. Therefore, of great importance is an optimal design of thermophysical experiments that is closely related to the method of inverse problem solution.

In view of the above said, the authors have tried to consider systematically the theoretical aspects of iterative regularization, to outline the ways of using this method in solution of applied problems, to present iterative algorithms for solving inverse heat transfer problems and the principles and algorithms for the design of thermophysical experiments.

Results given in the book were obtained by the authors. Section 1.6 was written by M. V. Klibanov, D.Sc.(Math) and the authors wish to express their gratitude to him. The structure and contents of the book can be inferred from the Contents List. Chapters 1 and 4 were written by O. M. Alifanov, Chapters 2, 3, and Appendix, by S. V. Rumyantsev, Chapters 5 and 6, by E. A. Artykhin.

Information for Readers. The following formula numbering system is adopted. The ordinal numbers of formulas, tables, and figures are given by the second digit. The first digit is the numbers of formulas refers to the Section number, in tables and figures it indicates the chapter number. The theorems and lemmas are numbered by one figure throughout each section, while referring to theorems and lemmas from another section, double numbering is used (the first figure refers to the section number where the theorem or lemma is presented).