APPENDIX B

THE THEORY OF SURFACES

We shall be concerned here with the fundamentals of the theory of surfaces that are necessary for understanding the material of Chapter 5. A more thorough knowledge of this theory can be obtained from the treatise [10.2].

Let the surface in question be described by the equation

$$\mathbf{r} = \mathbf{r}(\alpha^1, \alpha^2), \tag{B.1}$$

where α_1, α_2 are curvilinear coordinates. Further, let $\mathbf{n}(\alpha_1, \alpha_2)$ be a unit normal vector to the surface (see Figure B.1), and let the radius vector of an arbitrary point in the neighborhood of the surface be described by

$$\mathbf{R}(\alpha^1, \alpha^2; \xi) = \mathbf{r}(\alpha^1, \alpha^2) + \xi\mathbf{n}(\alpha^1, \alpha^2), \tag{B.2}$$

where ξ is the distance along the normal from the point to the surface. Such a coordinate system is said to be *normally connected with the surface.* All the essential relations in the newly introduced coordinate system may be obtained from the corresponding relations of Appendix A if we make substitutions $\alpha^3 = \xi$ and replace $\mathbf{R}$ by the expression (B.2).

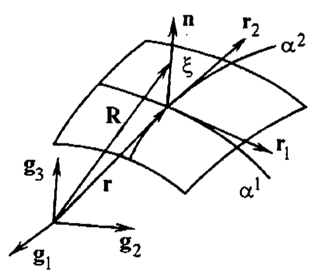

FIGURE B.1

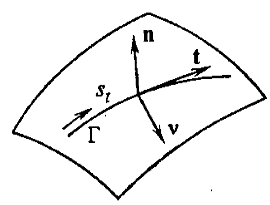

FIGURE B.2

Thus, formula (B.2) yields

$$\begin{gathered}\mathbf{R}_1 = \mathbf{r}_1 + \xi\mathbf{n}_1, \qquad \mathbf{R}_2 = \mathbf{r}_2 + \xi\mathbf{n}_2, \qquad \mathbf{R}_3 = \mathbf{n} \\ (\mathbf{r}_i = \partial\mathbf{r}/\partial\alpha^i,\ \mathbf{n}_i = \partial\mathbf{n}/\partial\alpha^i).\end{gathered} \tag{B.3}$$

In this and subsequent formulas letter indices are assumed to take the values 1,2. It follows from the expressions (B.3) and (A.11) that

$$g_{ij} = a_{ij} - \xi 2b_{ij} + \xi^2 \mathbf{n}_i \cdot \mathbf{n}_j , \qquad g_{13} = g_{23} = 0, \qquad g_{33} = 1, \tag{B.4}$$

where

$$\begin{aligned} a_{ij} &= \mathbf{r}_i \cdot \mathbf{r}_j = a_{ji}, \\ b_{ij} &= \mathbf{r}_i \cdot \mathbf{n}_j = \mathbf{n} \cdot \frac{\partial^2 \mathbf{r}_j}{\partial \alpha^j \, \partial \alpha^i} = \mathbf{r}_j \cdot \mathbf{n}_i = b_{ji}. \\ &(\mathbf{r}_i \cdot \mathbf{n} = 0) \end{aligned} \tag{B.5}$$

The material of this appendix is used in Chapter 5 for *thin shells*, in which the surface in question is the *middle* surface (being equidistant from the upper and the lower face surface). Therefore, only the closest proximity of the surface is taken into account, in which case

$$\xi \frac{b_{ij}}{\sqrt{a_{ii} a_{jj}}} \ll 1. \tag{B.6$_1$}$$

As is shown below, this inequality signifies that the shell thickness is small compared with the radii of curvature of the middle surface. This implies that all quantities to be considered are approximated by

$$\Psi(\alpha^1, \alpha^2 ; \xi) = \Psi^{(0)}(\alpha^1, \alpha^2) + \xi \Psi^{(1)}(\alpha^1, \alpha^2), \tag{B.6$_2$}$$

i.e., the dependence on ξ is assumed to be linear.

Thus, noting formulas (B.4), we immediately obtain

$$g_{ij} = a_{ij} - \xi^2 b_{ij} , \qquad g_{i3} = 0, \qquad g_{33} = 1 \tag{B.7}$$

and also see that $a_{ij} = g_{ij}|_{\xi=0}$ are the components of the symmetric metric tensor of the middle surface.

It follows from (A.16) that g^{ij} are the reduced minors of the elements g_{ij} in the determinant $g = |g_{ij}|$, which, in accord with (B.7), takes the form

$$g = \begin{vmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} = g_{11} g_{22} - g_{12} g_{21}. \tag{B.8}$$

Consequently

$$\begin{gathered} g^{11} = g_{22}/g , \qquad g^{12} = g^{21} = -g_{12}/g , \qquad g^{22} = g_{11}/g \\ (g^{13} = g^{31} = g^{23} = g^{32} = 0,\ g^{33} = 1). \end{gathered} \tag{B.9}$$

Recalling formulas (A.19), (A.26), (B.7), and (B.9), we have

$$ds^2 = (a_{\alpha\beta} - \xi b_{\alpha\beta})\, d\alpha^\alpha\, d\alpha^\beta, \tag{B.10}$$

$$\begin{aligned} &dS_i = \sqrt{gg^{ii}}\, d\alpha^i\, d\xi = \sqrt{g_{jj}}\, d\alpha^j\, d\xi \qquad (i \neq j), \\ &dS_3 = \sqrt{g}\, d\alpha^1\, d\alpha^2. \end{aligned} \tag{B.11}$$

Assuming in formulas (B.8),(B.9) that $\xi = 0$ and noting (B.7), we find that on the middle surface

$$a^{11} = a_{22}/a, \qquad a^{12} = a^{21} = -a_{12}/a, \qquad a^{22} = a_{11}/a, \tag{B.12}$$

$$a = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}. \tag{B.13}$$

Further, formulas (B.10)–(B.13) and (A.22) yield

$$ds_j = \sqrt{a_{jj}}\, d\alpha^j = \sqrt{aa^{ii}}\, d\alpha^j \qquad (i \neq j,\ i, j = 1, 2), \tag{B.14}$$

$$dS = dS_3|_{\xi=0} = \sqrt{a}\, d\alpha^1\, d\alpha^2, \tag{B.15}$$

$$\cos\chi = \cos\chi^{(3)}|_{\xi=0} = \frac{a_{12}}{\sqrt{a_{11}}\sqrt{a_{22}}}, \tag{B.16}$$

where ds_j is the length of an element of arc of the jth coordinate line of the middle surface, dS is the area of an element of the middle surface, and χ is the coordinate angle.

The relations (A.9), (A.11), and (A.15) on the middle surface take the form

$$\begin{gathered} \mathbf{r}_i \cdot \mathbf{r}_j = a_{ij}, \qquad \mathbf{r}^i \cdot \mathbf{r}^j = a^{ij}, \qquad \mathbf{r}_j \cdot \mathbf{r}^i = \delta^i_j, \\ \mathbf{r}_i = a_{i\alpha}\mathbf{r}^\alpha, \qquad \mathbf{r}^j = a^{j\beta}\mathbf{r}_\beta, \\ u_i = a_{i\alpha}u^\alpha, \qquad u^j = a^{j\beta}u_\beta, \\ t^{ij} = a^{i\alpha}t_\alpha^{\cdot j} = a^{j\beta}t^i_{\cdot\beta} = a^{i\alpha}a^{j\beta}t_{\alpha\beta}, \ldots. \end{gathered} \tag{B.17}$$

The surface discriminant tensor is introduced with the aid of the following relations [see formulas (A.23)–(A.25)]:

$$\begin{gathered} c_{ij} = \varepsilon_{ij3}|_{\xi=0} = \mathbf{n} \cdot (\mathbf{r}_i \times \mathbf{r}_j), \\ c^{ij} = \varepsilon^{ij3}|_{\xi=0} = \mathbf{n} \cdot (\mathbf{r}^i \times \mathbf{r}^j), \\ c_{12} = -c_{21} = \sqrt{a}, \qquad c_{11} = c_{22} = 0, \\ c^{12} = -c^{21} = 1/\sqrt{a}, \qquad c^{11} = c^{22} = 0; \end{gathered} \tag{B.18}$$

$$\begin{aligned} \mathbf{r}^i \times \mathbf{r}^j &= c_{ij}\mathbf{n}, \qquad & \mathbf{n} \times \mathbf{r}_j &= c_{j\alpha}\mathbf{r}^\alpha, \\ \mathbf{r}^i \times \mathbf{r}^j &= c^{ij}\mathbf{n}, \qquad & \mathbf{n} \times \mathbf{r}^j &= c^{j\alpha}\mathbf{r}_\alpha. \end{aligned} \tag{B.19}$$

It is seen from formulas (B.12) and (B.18)

$$a^{ij} = c^{i\alpha}c^{j\beta}a_{\alpha\beta}, \qquad a_{ij} = c_{i\alpha}c_{j\beta}a^{\alpha\beta}, \tag{B.20}$$

$$a^{i\beta}a_{j\beta} = \delta^i_j, \qquad a^{\alpha\beta}a_{\alpha\beta} = 2. \tag{B.21}$$

Let us now consider differentiation of coordinate vectors. According to formulas (A.32) and (A.31), we obtain

$$\begin{gathered} \Gamma^h_{ij} = \Gamma^h_{ji} = G^h_{ij}|_{\xi=0} = \frac{1}{2}\left(\frac{\partial a_{j\alpha}}{\partial \alpha^i} + \frac{\partial a_{i\alpha}}{\partial \alpha^j} - \frac{\partial a_{ij}}{\partial \alpha^\alpha}\right)a^{\alpha h}, \\ G^3_{ij}|_{\xi=0} = b_{ij}, \qquad G^h_{i3}|_{\xi=0} = b_{i\alpha}a^{\alpha h} = b^h_i, \\ G^h_{33} = G^3_{i3} = G^3_{33} = 0 \qquad (i, g, h = 1, 2); \end{gathered} \tag{B.22}$$

$$\begin{gathered} \frac{\partial \mathbf{r}_i}{\partial \alpha^j} = \Gamma^\alpha_{ij}\mathbf{r}_\alpha + b_{ij}\mathbf{n}, \qquad \frac{\partial \mathbf{r}^i}{\partial \alpha^j} = -\Gamma^i_{j\alpha}\mathbf{r}^\alpha + b^i_j\mathbf{n}, \\ \frac{\partial \mathbf{n}}{\partial \alpha^j} = -b_{j\alpha}\mathbf{r}^\alpha = -b^\alpha_j\mathbf{r}_\alpha. \end{gathered} \tag{B.23}$$

It is also seen from (B.22) and (B.12) that

$$\Gamma^\mu_{j\mu} = \frac{1}{2a}\frac{\partial a}{\partial \alpha^j} = \frac{1}{\sqrt{a}}\frac{\partial \sqrt{a}}{\partial \alpha^j}. \tag{B.24}$$

Let Γ be a line on the middle surface (see Figure B.2). It is linked to a triplet of unit vectors: a unit *tangential* vector $\mathbf{t}$ to the curve, a unit *normal* vector $\mathbf{n}$ to the surface, a unit *tangential normal* vector $\boldsymbol{\nu}$ (being the normal to the curve lying on the tangential plane). These unit vectors are related by

$$\begin{gathered} \mathbf{t} \times \mathbf{n} = \boldsymbol{\nu}, \qquad \mathbf{n} \times \boldsymbol{\nu} = \mathbf{t}, \qquad \boldsymbol{\nu} \times \mathbf{t} = \mathbf{n}, \\ \mathbf{t}\cdot\mathbf{t} = \mathbf{n}\cdot\mathbf{n} = \boldsymbol{\nu}\cdot\boldsymbol{\nu} = 1, \\ \mathbf{t}\cdot\mathbf{n} = \mathbf{n}\cdot\boldsymbol{\nu} = \boldsymbol{\nu}\cdot\mathbf{t} = 0. \end{gathered} \tag{B.25}$$

Besides, the following relation holds:

$$\mathbf{t} = \frac{d\mathbf{r}}{ds_t} = \frac{\partial \mathbf{r}}{\partial \alpha^\beta}\frac{d\alpha^\beta}{ds_t} = \frac{d\alpha^\beta}{ds_t}\mathbf{r}_\beta,$$

i.e.,

$$t^i = \frac{d\alpha^i}{ds_t}. \tag{B.26}$$

According to formulas (B.25) and (B.19) we have

$$\boldsymbol{\nu} = \mathbf{t} \times \mathbf{n} = t^\alpha \mathbf{r}_\alpha \times \mathbf{n} = c_{\beta\alpha} t^\alpha \mathbf{r}^\beta,$$

i.e.,

$$\nu_i = c_{i\gamma} t^\gamma. \tag{B.27}$$

Let us write the relations $(B.11)_2$ as the first of the following two expressions:

$$dS_i^\xi = \sqrt{g g^{ii}}\, d\alpha^j\, d\xi, \qquad dS_i = \sqrt{a a^{ii}}\, d\alpha^j\, d\xi \qquad (i \neq j), \tag{B.28}$$

a normal section of the shell being considered. It will be noted that the superscript ξ refers to the area of a surface element which is at a distance of ξ from the middle surface (see Figure B.3, a). Similarly, substituting $\boldsymbol{\nu}$ for $\mathbf{n}$ and noting equality (A.29), we get

$$\nu_i^\xi = \frac{1}{\sqrt{g^{ii}}} \frac{dS_i^\xi}{dS_\nu^\xi}, \qquad \nu_i = \frac{1}{\sqrt{a^{ii}}} \frac{dS_i}{dS_\nu}, \tag{B.29}$$

where

$$dS_\nu = ds_t\, d\xi \tag{B.30}$$

(see Figure B.3, b). The entities dS_ν^ξ, dS_ν are the areas of surface elements of the normal section passing through a tangent to the curve Γ, and ds_t is the length of the curve element.

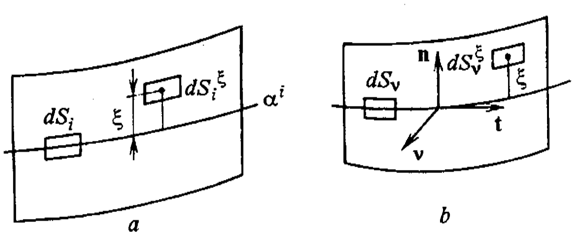

FIGURE B.3

In the *orthogonal* coordinate system associated with the surface

$$\mathbf{r}_i / \sqrt{a_{ii}} = \mathbf{r}^i \sqrt{a_{ii}} = \mathbf{e}_i, \tag{B.31}$$

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{n}$ are the unit coordinate vectors. The physical components of vectors and tensors associated with the surface are determined by the formulas

$$u_{(i)} = \sqrt{a_{ii}} u^i = u_i / \sqrt{a_{ii}}; \tag{B.32}$$

$$
\begin{aligned}
t_{(ij)} &= t_{ij}/\sqrt{a_{ii}a_{jj}} = \sqrt{a_{ii}a_{jj}}\,t^{ij} \\
&= \sqrt{a_{ii}}\,t^{i}_{\cdot j}/\sqrt{a_{jj}} = \sqrt{a_{ii}}\,t_{j}^{\cdot i}/\sqrt{a_{jj}}, \qquad \text{(B.33)} \\
t_{(i)n} &= t_{in}/\sqrt{a_{ii}} = \sqrt{a_{ii}}\,t^{i}_{\cdot n}.
\end{aligned}
$$

[cf. formula (A.43)].

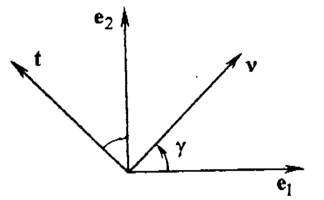

FIGURE B.4

Thus, noting Figure B.4, we find

$$
\begin{aligned}
\nu_{(1)} &= t_{(2)} = \cos\gamma, & \nu_{(2)} &= -t_{(1)} = -\sin\gamma, \\
\nu_1 &= \sqrt{a_{11}}\cos\gamma, & \nu_2 &= \sqrt{a_{22}}\sin\gamma, \\
t_1 &= -\sqrt{a_{11}}\sin\gamma, & t_2 &= \sqrt{a_{22}}\cos\gamma, \qquad \text{(B.34)} \\
\nu^1 &= \cos\gamma/\sqrt{a_{11}}, & \nu^2 &= \sin\gamma/\sqrt{a_{22}}, \\
t^1 &= -\sin\gamma/\sqrt{a_{11}}, & t^2 &= \cos\gamma/\sqrt{a_{22}}.
\end{aligned}
$$

Let

$$\mathbf{r}(\alpha^1,\alpha^2) = x_1(\alpha^1,\alpha^2)\mathbf{g}_1 + x_2(\alpha^1,\alpha^2)\mathbf{g}_2 + x_3(\alpha^1,\alpha^2)\mathbf{g}_3$$

be referred to a space rectangular Cartesian coordinate system with unit vectors $\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}_3$. First of all,

$$\mathbf{r}_i = \frac{\partial x_1}{\partial\alpha^i}\mathbf{g}_1 + \frac{\partial x_2}{\partial\alpha^i}\mathbf{g}_2 + \frac{\partial x_3}{\partial\alpha^i}\mathbf{g}_3,$$

and according to formulas (B.5)

$$a_{ij} = \frac{\partial x_1}{\partial\alpha^i}\frac{\partial x_1}{\partial\alpha^j} + \frac{\partial x_2}{\partial\alpha^i}\frac{\partial x_2}{\partial\alpha^j} + \frac{\partial x_3}{\partial\alpha^i}\frac{\partial x_3}{\partial\alpha^j}, \qquad \text{(B.35)}$$

$$\sin\chi\,\mathbf{n} = \frac{\mathbf{r}_1\times\mathbf{r}_2}{\sqrt{a_{11}}\sqrt{a_{22}}} = \mu_{13}\mathbf{g}_1 + \mu_{23}\mathbf{g}_2 + \mu_{33}\mathbf{g}_3, \qquad \text{(B.36)}$$

where

$$\begin{aligned}
\sqrt{a_{11}a_{22}}\mu_{13} &= \frac{\partial x_2}{\partial \alpha^1}\frac{\partial x_3}{\partial \alpha^2} - \frac{\partial x_3}{\partial \alpha^1}\frac{\partial x_2}{\partial \alpha^2}, \\
\sqrt{a_{11}a_{22}}\mu_{23} &= \frac{\partial x_3}{\partial \alpha^1}\frac{\partial x_1}{\partial \alpha^2} - \frac{\partial x_1}{\partial \alpha^1}\frac{\partial x_3}{\partial \alpha^2}, \\
\sqrt{a_{11}a_{22}}\mu_{33} &= \frac{\partial x_1}{\partial \alpha^1}\frac{\partial x_2}{\partial \alpha^2} - \frac{\partial x_2}{\partial \alpha^1}\frac{\partial x_1}{\partial \alpha^2}, \\
\sin\chi &= \sqrt{\mu_{13}^2 + \mu_{23}^2 + \mu_{33}^2},
\end{aligned} \tag{B.37}$$

and ξ is the coordinate angle. In view of the expressions (B.5) and (B.36), we obtain

$$\sin\chi\, b_{ij} = \mu_{13}\frac{\partial^2 x_1}{\partial \alpha^i\,\partial \alpha^j} + \mu_{23}\frac{\partial^2 x_2}{\partial \alpha^i\,\partial \alpha^j} + \mu_{33}\frac{\partial^2 x_3}{\partial \alpha^i\,\partial \alpha^j}. \tag{B.38}$$