Appendix A

Parametric Representation of Surfaces

A.1 Introduction

In many types of radiant analysis, the parametric method of representing surfaces has important advantages over the implicit method. In the implicit method, surfaces are represented by an equation of the form $f(x, y, z) = 0$. For example, the unit-radius sphere centered at the origin is represented implicitly as $x^2 + y^2 + z^2 - 1 = 0$ and the unit radius circular cylinder along the z-axis by $x^2 + y^2 - 1 = 0$.

A good starting point for parametric surface representation is a review of the parametric representation of curves, with which the student should already be familiar. The general expression for such a curve representation is

$$C: \qquad x = x(t); \qquad y = y(t); \qquad z = z(t) \qquad\qquad \alpha \leq t \leq \beta \tag{A.1}$$

where $x(t)$, $y(t)$,and $z(t)$ are three functions. For example, the unit radius circle in the horizontal plane one unit up the z-axis and centered along the z-axis is given by

$$C: \qquad x = \cos(t); \qquad y = \sin(t); \qquad z = 1 \qquad\qquad 0 \leq t \leq 2\pi \tag{A.2}$$

As parameter t takes on all possible values in the interval $\alpha \leq t \leq \beta$, the point at coordinates (x, y, z) covers the entire curve. Another example is the parametric representation of the straight line connecting the origin to the point $(1, 2, 3)$:

$$C: \qquad x = t; \qquad y = 2t; \qquad z = 3t \qquad\qquad 0 \leq t \leq 1 \tag{A.3}$$

Curves can also be defined parametrically in vector form, for which the general representation is

$$C: \qquad \mathbf{r} = x(t)\widehat{\mathbf{i}} + y(t)\widehat{\mathbf{j}} + z(t)\widehat{\mathbf{k}} \qquad\qquad \alpha \leq t \leq \beta \tag{A.4}$$

(Here, $\mathbf{r}$ is of course the position vector, $= (x, y, z)$, and $\widehat{\mathbf{i}}$, $\widehat{\mathbf{j}}$, and $\widehat{\mathbf{k}}$ are the unit vectors along the x-, y-, and z- axis, respectively.) In this form, the curve C has the following interpretation: let the tail of the position vector be kept at the origin; then as t runs continuously through all values from α to β, the tip of the position vector traces out the curve C in space. Note that the general curve can now be given by the compact expression:

$$C: \qquad \mathbf{r} = \mathbf{r}(t) \qquad \alpha \leq t \leq \beta \tag{A.5}$$

where $\mathbf{r}(t) = x(t)\widehat{\mathbf{i}} + y(t)\widehat{\mathbf{j}} + z(t)\widehat{\mathbf{k}}$, is a vector function of t.

The properties of the curve, such as its elemental length ds, its tangent vector $\widehat{\mathbf{T}}$, and its normals $\widehat{\mathbf{N}}$ and $\widehat{\mathbf{B}}$, can be found by straightforward operations on $\mathbf{r}(t)$. For example

$$ds = J(t)dt \quad \text{where} \quad J(t) = |d\mathbf{r}(t)/dt| \tag{A.6}$$

and $\widehat{\mathbf{T}} = [d\mathbf{r}(t)/dt]/J(t)$. The curve's length L is given by

$$L = \int_{\alpha}^{\beta} J(t)dt \tag{A.7}$$

It should be recognized that the parametric representation of a curve is not unique; that is, there is more than one function $\mathbf{r}(t)$ that can specify the same curve in space. For example, the parametric representation

$$C: \qquad \mathbf{r} = \frac{1-t^2}{1+t^2}\widehat{\mathbf{i}} + \frac{2t}{1+t^2}\widehat{\mathbf{j}} + \widehat{\mathbf{k}} \qquad 0 \leq t \leq 1 \tag{A.8}$$

represents the same circle as that described by Eq. (A.2).

The parametric equation for a surface is a simple extension of the equation for a curve. The general representation corresponding to Eq. (A.1) is

$$S: \qquad x = x(u,v); \quad y = y(u,v); \quad z = z(u,v) \qquad u, v \in R_{u,v} \tag{A.9}$$

where $x(u, v)$, $y(u, v)$, and $z(u, v)$ are functions of u and v. Thus, in this representation, there are two parameters, u and v, instead of the single parameter t. As u and v run through all the values inside a prescribed range, the point at coordinate (x, y, z) moves through values that cover the entire surface. For every combination of u and v, there will be one point on the surface. It is useful to conceptually construct a u-v plane, having u along the ordinate axis and v along the abscissa axis. Then we can say that for every point on the surface, there is a unique corresponding point on the u-v plane and vice-versa.

The prescribed range of u and v is defined by the statement $u, v \in R_{u,v}$, which corresponds to the statement $\alpha \leq t \leq \beta$ in Eq. (A.1). The $R_{u,v}$ defines the region in the u-v plane over which u and v are allowed to range, and the symbol $\in$ can be interpreted as "are contained within." The simplest region $R_{u,v}$ is a rectangular region bounded by lines of constant u and v in the $u - v$

plane. This would be covered by the statements: $\alpha_u \le u \le \beta_u$, $\alpha_v \le v \le \beta_v$, and if this applies, then $u, v \in R_{u,v}$ in Eq. (A.9) would simply be replaced by these two statements.

A few examples should clarify the above. An $L \times W$ rectangle in a vertical plane h unit along the positive x-axis would be represented by

$$S: \quad x = h; \quad y = Lu; \quad z = Wv \qquad 0 \le u \le 1, 0 \le v \le 1 \tag{A.10}$$

and the circle of radius R with axis along the z-axis and center h units up that axis is represented by

$$S: \quad x = Rv\cos(2\pi u); \quad y = Rv\sin(2\pi u); \quad z = h \qquad 0 \le u \le 1, 0 \le v \le 1 \tag{A.11}$$

Note that in this last representation, it is the *complete* circle that is being represented; not just the outside arc, which is a curve rather than a surface. As v takes on different values, various circular curves are represented, and the totality of these curves is the complete circle.

As was the case for curves, parametric-surface representation is not unique. We will give the name "normalized" to those surface representations that, like Eqs. (A.10) and (A.11), have the region $R_{u,v}$ given by the square $0 \le u \le 1$, $0 \le v \le 1$. Indeed, by judicious choice of the functions $x(u,v)$, $y(u,v)$, and $z(u,v)$ it is possible to represent a very wide assortment of surfaces in this way. For example, the NURBS surface representations widely used in commercial CADCAM programs are of this type. Unless specifically stated otherwise, we will restrict our attention to normalized surface representations in this book, and from now on, the statement of the region $R_{u,v}$ will be deleted as being understood to be $0 \le u \le 1, \;\; 0 \le v \le 1$.

Surfaces can also be expressed in vector form, for which the general representation is

$$S: \qquad \mathbf{r} = x(u,v)\widehat{\mathbf{i}} + y(u,v)\widehat{\mathbf{j}} + z(u,v)\widehat{\mathbf{k}} \qquad \text{or} \qquad \mathbf{r} = \mathbf{r}(u,v) \tag{A.12}$$

where $\mathbf{r}(u,v) = x(u,v)\widehat{\mathbf{i}} + y(u,v)\widehat{\mathbf{j}} + z(u,v)\widehat{\mathbf{k}}$ is a vector function of u and v. In this form, the surface S has the following interpretation: let the tail of the position vector be kept at the origin; then as u and v run continuously over $R_{u,v}$, the tip of the position vector carves out the surface S in space. For example, the circle of Eq. A.11 can be written as $\mathbf{r} = \mathbf{r}_c + Rv\cos(2\pi u)\widehat{\mathbf{i}} + Rv\sin(2\pi u)\widehat{\mathbf{j}}$, where $\mathbf{r}_c = (0, 0, h)$. Note that the general surface can now be given by the compact expression: $\mathbf{r} = \mathbf{r}(u,v)$.

As with curves, the various properties of the surfaces can be readily expressed. One can show that the vector $\mathbf{J}(u,v)$ given by

$$\mathbf{J}(u,v) = \frac{\partial(y,z)}{\partial(u,v)}\widehat{\mathbf{i}} + \frac{\partial(z,x)}{\partial(u,v)}\widehat{\mathbf{j}} + \frac{\partial(x,y)}{\partial(u,v)}\widehat{\mathbf{k}} \tag{A.13}$$

is always normal to the surface. In Eq. (A.13), terms like $\frac{\partial(y,z)}{\partial(u,v)}$ (called Jacobians) have a special meaning:

$$\frac{\partial(F,G)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix} \tag{A.14}$$

in which each of F and G can be $x(u.v)$ or $y(u,v)$ or $z(u,v)$. Therefore, a unit normal is

$$\hat{\mathbf{n}}(u,v) = \pm\mathbf{J}(u,v)/\left|\mathbf{J}(u,v)\right| \tag{A.15}$$

(The choice of the plus or minus sign will be discussed later.) The area dA_S cut out on the surface when u goes from u to $u + du$ and v goes from v to $v + dv$ can be shown to be given by

$$dA_S = \left|\mathbf{J}(u,v)\right| dudv = J(u,v)dudv \tag{A.16}$$

where $J(u,v) = |\mathbf{J}(u,v)|$, is called here the "surface factor." The area A_S of surface S is therefore given by

$$A_S = \int_0^1 \int_0^1 J(u,v)dudv \tag{A.17}$$

In radiant analysis, it is important to recognize that every surface of practical interest has in fact two sides. For example, the rectangle of Eq. (A.10) has one side facing in the positive x-direction and another facing the negative x-direction. And the circle of Eq. (A.11) has one side facing in the positive z-direction and another facing the negative z-direction. If there were a small source of radiant energy near the origin, it would irradiate only one side of each of these two surfaces. It is clearly necessary to specify to which "surface side" we are referring. It is to be noted that both surface sides appear to have the same mathematical representation: $\mathbf{r} = \mathbf{r}(u,v)$. They also have the same $J(u,v)$ and area.

But they will not be the same $\hat{\mathbf{n}}(u,v)$ because the mathematical distinction between the two surface sides will be carried in the normal. The convention throughout this book will be that when located with its tip on the surface, the normal points *into* the surface side of interest. That is, if we were to place our finger on the surface side in question and then press it into the surface, $\hat{\mathbf{n}}(u,v)$ would be in the direction of the pressing force. For example, suppose of the two surface sides represented by Eq. (A.11), we want to represent the one that faces the origin. Then we could specify $\hat{\mathbf{n}}(u,v) = +\widehat{\mathbf{k}}$. And if we wanted to represent the surface side facing away from the origin, we could specify $\hat{\mathbf{n}}(u,v) = -\widehat{\mathbf{k}}$.

Thus, the full specification of a surface side will require its parametric surface representation $\mathbf{r} = \mathbf{r}(u,v)$, *plus* the specification of the normal. The specification of the normal can be carried in words or in a mathematical statement like $\hat{\mathbf{n}}(u,v) = -\widehat{\mathbf{k}}$. Because of the great diversity of surfaces, no single method of specifying the normal in words works for every configuration, but the normal specification, once made, will automatically fix the choice of plus or minus sign in Eq. (A.15).

A.2 Catalog of Surface Representations

The following is a list of suitable, normalized parametric representations of various surface sides of interest for engineering radiant analysis. We also include below the properties of the surface, forestalling the need for the user of the catalog to apply Eqs. (A.14)–(A.17). The user must, however, choose the plus or minus sign on $\hat{\mathbf{n}}(u, v)$ to indicate the surface side wanted. This choice can be usually achieved by inspection. Where the choice may be difficult, guidelines are given.

In several entries, the representation contains reference to a unit vector $\hat{\mathbf{m}}$, that is stated to be any unit vector that is normal to another specified unit vector, say $\hat{\mathbf{n}}$. The role of $\hat{\mathbf{m}}$ is generally to fix an arbitrary reference direction from which angles are measured. Ordinarily, a suitable choice for $\hat{\mathbf{m}}$ is $(\hat{\mathbf{i}} \times \hat{\mathbf{n}} / \left|\hat{\mathbf{i}} \times \hat{\mathbf{n}}\right|)$, but this choice is not suitable if $\hat{\mathbf{n}}$ happens to equal $\hat{\mathbf{i}}$; in this case one should choose $(\hat{\mathbf{j}} \times \hat{\mathbf{n}} / \left|\hat{\mathbf{j}} \times \hat{\mathbf{n}}\right|)$.

1. Rectangle

Let $\mathbf{r}_A, \mathbf{r}_B$, and $\mathbf{r}_C$ be the position vectors of any three vertices of the rectangle such that $\mathbf{r}_B$ and $\mathbf{r}_C$ are at opposite corners. Then

$$\mathbf{r} = \mathbf{r}_A + (\mathbf{r}_B - \mathbf{r}_A)u + (\mathbf{r}_C - \mathbf{r}_A)v$$

$$\hat{\mathbf{n}}(u, v) = \pm \frac{(\mathbf{r}_A - \mathbf{r}_B) \times (\mathbf{r}_A - \mathbf{r}_C)}{|(\mathbf{r}_A - \mathbf{r}_B) \times (\mathbf{r}_A - \mathbf{r}_C)|}$$

$$J(u, v) = |\mathbf{r}_A - \mathbf{r}_B| \cdot |\mathbf{r}_A - \mathbf{r}_C|\,; \qquad A_S = |\mathbf{r}_A - \mathbf{r}_B| \cdot |\mathbf{r}_A - \mathbf{r}_C|$$

2. Circle and Circle Sector

Let $\mathbf{r}_c$ be the the position vector of the center of the circle, R be its radius, and $\hat{\mathbf{n}}$ be the unit vector in direction of the desired normal to the circle. Then for the circle,

$$\mathbf{r} = \mathbf{r}_c + Ru\,[\hat{\mathbf{m}}\cos(2\pi v) + \hat{\mathbf{m}} \times \hat{\mathbf{n}}\sin(2\pi v)]$$

$$\hat{\mathbf{n}}(u, v) = \hat{\mathbf{n}}; \quad J(u, v) = 2\pi u R^2; \qquad A_S = \pi R^2$$

where $\hat{\mathbf{m}}$ is any unit vector perpendicular to $\hat{\mathbf{n}}$.

For the sector of a circle, everywhere replace π by $\theta/2$, where θ is the angle of the sector and make $\hat{\mathbf{m}}$ lie along one of the radial arms of the sector.

3. Circular Cylinder (curved part only)

Let R be the cylinder's radius and L be its axial length. Also, let $\hat{\mathbf{a}}$ be the unit vector in direction of the cylinder's axis (pointing away from the base), and let $\mathbf{r}_c$ be the position vector of the center of the circle at the cylinder's base. Then the curved part of the cylinder is represented by

$$\mathbf{r} = \mathbf{r}_c + R\,[\hat{\mathbf{m}}\cos(2\pi v) + \hat{\mathbf{m}} \times \hat{\mathbf{a}}\sin(2\pi v)] + \hat{\mathbf{a}}Lu$$

$$\hat{\mathbf{n}}(u, v) = \pm\,[\hat{\mathbf{m}}\cos(2\pi v) + \hat{\mathbf{m}} \times \hat{\mathbf{a}}\sin(2\pi v)]$$

$$J(u,v) = 2\pi RL; \qquad A_S = 2\pi RL$$

where $\hat{\mathbf{m}}$ is any unit vector perpendicular to $\hat{\mathbf{a}}$.

In the expression for $\hat{\mathbf{n}}(u,v)$, choose the plus sign if the cylinder's inside surface is the surface side of interest; choose the negative sign if the cylinder's outside surface is the surface side of interest.

4. Sphere

Let the position vector of the center of the sphere be $\mathbf{r}_c$ and let R be the radius of the sphere. Then

$$\mathbf{r} = \mathbf{r}_c + R\left[\cos(2\pi u)\sin(\pi v)\widehat{\mathbf{i}} + \sin(2\pi u)\sin(\pi v)\widehat{\mathbf{j}} + \cos(\pi v)\widehat{\mathbf{k}}\right]$$

$$\hat{\mathbf{n}}(u,v) = \pm\left[\cos(2\pi u)\sin(\pi v)\widehat{\mathbf{i}} + \sin(2\pi u)\sin(\pi v)\widehat{\mathbf{j}} + \cos(\pi v)\widehat{\mathbf{k}}\right]$$

$$J(u,v) = 2\pi^2\sin(\pi v)R^2; \qquad A_S = 4\pi R^2$$

In the expression for $\hat{\mathbf{n}}(u,v)$, choose the plus sign if the sphere's inside surface is the surface side of interest; choose the negative sign if the sphere's outside surface is the surface side of interest.

5. Triangle

Let the position vectors of the three vertices of the triangle be $\mathbf{r}_A$, $\mathbf{r}_B$, and $\mathbf{r}_C$, respectively. Then

$$\mathbf{r} = (1-u)\mathbf{r}_A + u\mathbf{r}_B + (\mathbf{r}_C - \mathbf{r}_B)uv$$

$$\hat{\mathbf{n}}(u,v) = \pm\frac{(\mathbf{r}_A - \mathbf{r}_B)\times(\mathbf{r}_A - \mathbf{r}_C)}{|(\mathbf{r}_A - \mathbf{r}_B)\times(\mathbf{r}_A - \mathbf{r}_C)|}$$

$$J(u,v) = |(\mathbf{r}_A - \mathbf{r}_B)\times(\mathbf{r}_A - \mathbf{r}_C)|\,u; \qquad A_S = |(\mathbf{r}_A - \mathbf{r}_B)\times(\mathbf{r}_A - \mathbf{r}_C)|/2$$

6. Circular Annulus and Sector of a Circular Annulus

Let R_o be the radius of the annulus's outer circle and R_i be the radius of the annulus's inner circle. Also, let $\mathbf{r}_1$ be the position vector of the center of the complete circle of which the annulus is a part, and let $\hat{\mathbf{n}}$ be the unit vector in direction of desired normal to the annulus. Then, for the annulus

$$\mathbf{r} = \mathbf{r}_1 + [R_i + (R_o - R_i)u]\,[\hat{\mathbf{m}}\cos(2\pi v) + \hat{\mathbf{m}}\times\hat{\mathbf{n}}\sin(2\pi v)]$$

$$\hat{\mathbf{n}}(u,v) = \hat{\mathbf{n}}; \quad J(u,v) = 2\pi\,[R_i + (R_o - R_i)u]\,(R_o - R_i); \qquad A_S = \pi(R_o^2 - R_i^2)$$

where $\hat{\mathbf{m}}$ is any unit vector perpendicular to $\hat{\mathbf{n}}$.

For the sector of an annulus, everywhere replace π by $\theta/2$, where θ is the angle of the sector, and make $\hat{\mathbf{m}}$ lie along one of the radial arms of the sector.

7. Known Shape Moved to Another Position and Orientation

Let a surface that originally encloses the origin O be translated and rotated to some new position in space. Suppose the parametric representation of the surface when its in the original position is known to be

$$\mathbf{r} = \mathbf{r}_{\text{old}}(u,v) = x_{\text{old}}(u,v)\widehat{\mathbf{i}} + y_{\text{old}}(u,v)\widehat{\mathbf{j}} + z_{\text{old}}(u,v)\widehat{\mathbf{k}}$$

$$\hat{\mathbf{n}}(u,v) = \hat{\mathbf{n}}_{\text{old}}(u,v) \qquad J(u,v) = J_{\text{old}}(u,v) \qquad A_S = A_{S\text{old}}$$

and we want to find its representation in its new position. Let us say that when the surface is in its original position, it intersects the x-axis at location X and the y-axis at location Y. Let the point on or inside the surface that was at O have position vector $\mathbf{r}_{\text{O}}$ after translation and let the position vectors of the new positions of X and Y be $\mathbf{r}_{\text{X}}$ and $\mathbf{r}_{\text{Y}}$, respectively. Then the parametric representation of the surface in its new position is given by

$$\mathbf{r} = \mathbf{r}_{\text{O}} + x_{\text{old}}(u,v)\hat{\mathbf{m}} + y_{\text{old}}(u,v)\hat{\mathbf{p}} + z_{\text{old}}(u,v)\hat{\mathbf{q}}$$

$$\hat{\mathbf{n}}(u,v) = \hat{\mathbf{n}}_{\text{old}}(u,v)\cdot(m_x,p_x,q_x)\widehat{\mathbf{i}} + \hat{\mathbf{n}}_{\text{old}}(u,v)\cdot(m_y,p_y,q_y)\widehat{\mathbf{j}} + \hat{\mathbf{n}}_{\text{old}}(u,v)\cdot(m_z,p_z,q_z)\widehat{\mathbf{k}}$$

$$J(u,v) = J_{\text{old}}(u,v); \qquad A_S = A_{S\text{old}}$$

where unit vectors $\hat{\mathbf{m}} = (m_x, m_y, m_z)$, $\hat{\mathbf{p}} = (p_x, p_y, p_z)$ and $\hat{\mathbf{q}} = (q_x, q_y, q_z)$ are given by

$$\hat{\mathbf{m}} = (\mathbf{r}_{\text{O}} - \mathbf{r}_{\text{X}})/\left|\mathbf{r}_{\text{O}} - \mathbf{r}_{\text{X}}\right|; \quad \hat{\mathbf{p}} = (\mathbf{r}_{\text{O}} - \mathbf{r}_{\text{Y}})/\left|\mathbf{r}_{\text{O}} - \mathbf{r}_{\text{Y}}\right|; \quad \text{and} \quad \hat{\mathbf{q}} = \hat{\mathbf{m}} \times \hat{\mathbf{p}}.$$

8. Ellipse

Let a and b be the ellipses's major and minor axes. Let the ellipse be situated on the $x - y$ plane, centered at the origin. Then

$$\mathbf{r} = au\cos(2\pi v)\widehat{\mathbf{i}} + bu\sin(2\pi v)\widehat{\mathbf{j}}$$

$$\hat{\mathbf{n}}(u,v) = \pm\widehat{\mathbf{k}}; \qquad J(u,v) = 2\pi abu; \qquad A_S = \pi ab$$

Use Entry 7 above if the ellipse is in a different position.

9. Hemisphere and Other Parts of a Sphere

Let R be the hemisphere's radius and let it be situated with its circular base on the x-y plane with the center of the base at the origin. Then

$$\mathbf{r} = R\cos(2\pi u)\sin(\pi v/2)\widehat{\mathbf{i}} + R\sin(2\pi u)\sin(\pi v/2)\widehat{\mathbf{j}} + R\cos(\pi v/2)\widehat{\mathbf{k}}$$

$$\hat{\mathbf{n}}(u,v) = \pm\left[\cos(2\pi u)\sin(\pi v/2)\widehat{\mathbf{i}} + \sin(2\pi u)\sin(\pi v/2)\widehat{\mathbf{j}} + \cos(\pi v/2)\widehat{\mathbf{k}}\right]$$

$$J(u,v) = \pi^2\sin(\pi v/2)R^2; \qquad A_S = 2\pi R^2$$

Use Entry 7 if the hemisphere is in a different position. By replacing $2\pi u$ by θu and/or πv by φv, one can generate various other parts of the sphere by letting θ and φ take on various values.

10. Truncated Right Circular Cone (Curved Surface)

Let the cone's circular base be in the x-y plane, centered at the origin. Let R_b be the radius of the base, R_t be the radius of the top, and h be its height. Then the curved surface of the cone is given by

$$\mathbf{r} = R_b\left(1 - u\frac{R_b - R_t}{R_b}\right)\left[\widehat{\mathbf{i}}\cos(2\pi v) + \widehat{\mathbf{j}}\sin(2\pi v)\right] + hu\widehat{\mathbf{k}}$$

$$\hat{\mathbf{n}}(u,v) = \pm\left[h\cos(2\pi v)\widehat{\mathbf{i}} + h\sin(2\pi v)\widehat{\mathbf{j}} + (R_b - R_t)\widehat{\mathbf{k}}\right]/\sqrt{h^2 + (R_b - R_t)^2}$$

$$J(u,v) = 2\pi R_b\sqrt{h^2 + (R_b - R_t)^2}\left(1 - u\frac{R_b - R_t}{R_b}\right);$$

$$A_S = \pi(R_b + R_t)\sqrt{h^2 + (R_b - R_t)^2}$$

Choose the plus sign in the $\hat{\mathbf{n}}(u,v)$ expression to represent the inside of the cone; choose the minus sign for the outside of the cone. Use Entry 7 above if the cone is in a different position.

11. Torus

Let the torus have its axis along the z-axis and be centered at the origin. That is, the intersection of the torus with the x-y plane consists of two circles with centers at the origin. The radius of the inner circle is $a - b$ (with $b > a$). The radius of the outer circle is $a + b$. Then the surface of the torus is given by

$$\mathbf{r} = [a + b\cos(2\pi u)]\left[\cos(2\pi v)\widehat{\mathbf{i}} + \sin(2\pi v)\widehat{\mathbf{j}}\right] + b\sin(2\pi u)\widehat{\mathbf{k}}$$

$$\hat{\mathbf{n}}(u,v) = \pm\left[\cos(2\pi v)\cos(2\pi u)\widehat{\mathbf{i}} + \cos(2\pi u)\sin(2\pi v)\widehat{\mathbf{j}} + \sin(2\pi u)\widehat{\mathbf{k}}\right]$$

$$J(u,v) = 4\pi^2 b\,[a + b\cos(2\pi u)]\,; \qquad A_S = 4\pi^2 ab$$

Choose the plus sign in the $\hat{\mathbf{n}}(u,v)$ expression to represent the inside of the torus; choose the minus sign for the outside of the torus. Use Entry 7 if the torus is in a different position.

A.3 Application to Radiant Analysis

A.3.1 Solid Angle Evaluation

Equation 3.7 gives the formula for the evaluation of a solid angle subtended by a body from some point P in space:

$$\omega_{object} = \iint_S \frac{\hat{\mathbf{e}}(x,y,z)\cdot\hat{\mathbf{n}}(x,y,z)}{r(x,y,z)^2}dA \tag{A.18}$$

For convenience, let the point P have position vector $\mathbf{r}_P$ and let the object's surface be represented parametrically by $\mathbf{r} = \mathbf{r}(u, v)$. The parametric representation means that we can replace (x, y, z) by (u, v) to represent functional dependence of a point on the surface; thus $\hat{\mathbf{e}}(x, y, z) \to \hat{\mathbf{e}}(u, v)$ and $r(x, y, z) \to r(u, v)$. Also, $r(u, v) = |\mathbf{r}(u, v) - \mathbf{r}_P|$ and $\hat{\mathbf{e}}(u, v) = (\mathbf{r}(u, v) - \mathbf{r}_P) / |\mathbf{r}(u, v) - \mathbf{r}_P|$, and since $dA = J(u, v)dudv$, we obtain

$$\omega_{object} = \int_0^1 \int_0^1 \frac{\hat{\mathbf{e}}(u, v) \cdot \hat{\mathbf{n}}(u, v)}{|\mathbf{r}(u, v) - \mathbf{r}_P|^2} J(u, v)dudv \tag{A.19}$$

Figure A.1 shows a Mathcad worksheet of a solid angle calculation, this for a circle side centered at coordinate (1,2,3) and directed with its normal along (1,1,1); the solid angle calculated is that subtended from the point P having coordinate (2,1,1). Such calculations can be readily performed using any of the surfaces given in the above catalog, or indeed any other having a suitable parametric representation.

Before proceeding further, however, we need to note that, as written, Eq. (A.19) would yield negative values for ω_{object} for surface sides that face *away* from point P. This is because the angle between the normal $\hat{\mathbf{n}}(u, v)$ and $\hat{\mathbf{e}}(u, v)$ becomes greater than $\pi/2$ for these surface sides. We consider this negative answer to be an unrealistic result, and adopt the following convention: surface sides that face away from the point P subtend a *zero* solid angle from P. Moreover, when surface sides face P over part of their area and face away from P over other parts, the parts that face away from P contribute nothing to the integral of Eqs. (A.18) and (A.19). While this rule may appear to be simply a mathematical convention, in fact it is just what is needed to make physical sense when similar calculations are made in radiant analysis.

To enforce this rule, Eq. (A.19) needs to be changed, and the needed change can achieved be by the expedient of introducing the "positive-only" function, which will be symbolized here by poso(x), defined as follows:

$$\text{poso}(x) = \frac{x + |x|}{2} \qquad \text{or} \qquad \text{poso}(x) = x\text{step}(x) \tag{A.20}$$

where step is the Heaviside Step function. Either of these definitions make

$$\text{poso}(x) = x \text{ when } x > 0; \quad \text{poso}(x) = 0 \text{ when } x \leq 0 \tag{A.21}$$

and both are readily programmed. So the proper expression for ω_{object} is

$$\omega_{object} = \int_0^1 \int_0^1 \frac{\text{poso}\left(\hat{\mathbf{e}}(u, v) \cdot \hat{\mathbf{n}}(u, v)\right)}{|\mathbf{r}(u, v) - \mathbf{r}_P|^2} J(u, v)dudv \tag{A.22}$$

Inspection of the Mathcad worksheet given in Fig. A.1 shows that the poso function has already been incorporated there.

An example of a surface side that faces the point P in some parts and faces away from P in other parts is the outside surface of a sphere when the point P is outside the sphere. The points on the sphere nearest P do face P, while the points on the other side face away from it. The dividing line between these sets of points is a circle on the sphere that contains all the points for which a line running from the point to P is tangent to the sphere. By elementary geometric calculations, one can show that the solid angle subtended by a sphere of radius R and with center distance r_{cc} from P is given by $\omega_{object} = 2\pi(1-\sqrt{1-R^2/r_{cc}^2})$. The student is encouraged to evaluate ω_{object} for spheres using both this formula and by Mathcad, replacing the circle representation used in Fig. A.1 to a suitable parametric representation for the sphere.

A.3.2 Point Form-Factor Evaluation

In Eq. (5.18), the point form factor to surface A_j from some elemental area dA_1 with inward normal $\hat{\mathbf{n}}_1$ and located at $\mathbf{r}_1$ is given by Eq. (5.20):

$$F_{d1-j} = \iint_{S_j} \frac{\{\mathbf{n}_1 \cdot [-\hat{\mathbf{e}}(\mathbf{r}_j, \mathbf{r}_1)]\} \cdot [\hat{\mathbf{e}}(\mathbf{r}_j, \mathbf{r}_1) \cdot \hat{\mathbf{n}}(\mathbf{r}_j)]}{\pi \left|\mathbf{r}_j - \mathbf{r}_1\right|^2} dA_j$$

Reexpressing this in terms of the parametric representation gives

$$F_{d1-j} = \int_0^1 \int_0^1 \frac{\mathrm{poso}\,(\hat{\mathbf{n}}_1 \cdot [-\hat{\mathbf{e}}(u,v)]) \cdot \mathrm{poso}\,[\hat{\mathbf{e}}(u,v) \cdot \hat{\mathbf{n}}(u,v)]}{\pi \left|\mathbf{r}(u,v) - \mathbf{r}_1\right|^2} J(u,v)dudv \tag{A.23}$$

Note that we have also introduced the poso function in Eq. (A.23); with $\hat{\mathbf{n}}_1 \cdot [-\hat{\mathbf{e}}(u,v)]$ as argument, this will ensure that if some part of A_j falls behind dA_1, then it does not contribute to the radiant flux on dA_1.

Point form factors can, therefore, be determined in the same way as solid angles. The only additional piece of information that has to be provided is the direction of the normal $\hat{\mathbf{n}}_1$. Figure A.1 includes in its Addendum 1 the calculation of the point form factor for $\mathbf{n}_1 = (0,0,1)$, and for A_j being the same surface as had been used in the previous part of the figure. Similar calculations can clearly be made for other surfaces and for other locations and orientations of dA_1.

A.3.3 Gaseous Point Form-Factor Evaluation

In Eq. (5.24), the gaseous point form factor to surface A_j from some elemental area dA_1 with inward normal $\hat{\mathbf{n}}_1$ and located at $\mathbf{r}_1$ is given by the surface integral

$$G_{d1-j}(a) = \iint_{S_j} \frac{\exp(-ar(x,y,z))\,\{\hat{\mathbf{n}}_1 \cdot [-\hat{\mathbf{e}}(x,y,z)]\} \cdot [\hat{\mathbf{e}}(x,y,z) \cdot \hat{\mathbf{n}}(x,y,z)]}{\pi r(x,y,z)^2} dA \tag{A.24}$$

Reexpressing in terms of the parametric representation gives

$$G_{d1-j}(a) = \int_0^1 \int_0^1 \frac{e^{-a|\mathbf{r}(u,v)-\mathbf{r}_1|}\mathrm{poso}(\hat{\mathbf{n}}_1 \cdot [-\hat{\mathbf{e}}(u,v)]) \cdot \mathrm{poso}\,(\hat{\mathbf{e}}(u,v) \cdot \hat{\mathbf{n}}(u,v))}{\pi \left|\mathbf{r}(u,v) - \mathbf{r}_1\right|^2} J(u,v)dudv \tag{A.25}$$

Thus, for any specified value of a, gaseous point form factors can be determined in the same way that F_{d1-j} was calculated in the previous section. Figure A.1 includes in its Addendum 2 the calculation of the gaseous point form factor for $a = 0.5$, for the same geometric circumstances as before. Similar calculations can clearly be made for other surfaces, other values of a, and other locations and orientations of dA_1.

A.3.4 Form-Factor Evaluation

Let the two surfaces involved in the form factor be denoted P and Q, represented by subscripts p and q, respectively. Then from Eq. (5.30), we have

$$F_{p-q} = \frac{1}{A_p}\iint_{S_p}\iint_{S_q} \frac{\operatorname{poso}(\hat{\mathbf{n}}_p(\mathbf{r}_p)\cdot[-\hat{\mathbf{e}}(\mathbf{r}_p,\mathbf{r}_q)])\cdot\operatorname{poso}(\hat{\mathbf{e}}(\mathbf{r}_p,\mathbf{r}_q)\cdot\hat{\mathbf{n}}_q(\mathbf{r}_q))}{\pi\,|\mathbf{r}_p-\mathbf{r}_q|^2}\,dA_p dA_q \tag{A.26}$$

We will need parameters u and v for each surface. Let the parameters for surface P be u_p and v_p, and those for Q be $u_q$and v_q. Further, to arrive at compact expressions, we adopt the following conventions:

1. Let $\mathbf{u}$ represent the two component vector (u, v); that is, $\mathbf{u}$ is the position vector of a point in the u-v plane. Thus, we have $\mathbf{u}_p$=(u_p, v_p) for surface P and $\mathbf{u}_q$=(u_q, v_q) for surface Q.
2. Let surface P be represented parametrically by $\mathbf{r} = \mathbf{r}_p(u_p, v_p) = \mathbf{r}_p(\mathbf{u}_p)$, and Q be represented parametrically by $\mathbf{r} = \mathbf{r}_q(u_q, v_q) = \mathbf{r}_q(\mathbf{u}_q)$.
3. Let the area-factor $J(\mathbf{u})$ be $J_p(\mathbf{u}_p)$ for P and $J_q(\mathbf{u}_q)$ for Q, and the local unit normals be $\hat{\mathbf{n}}_p(\mathbf{u}_p)$ for P and $\hat{\mathbf{n}}_q(\mathbf{u}_q)$ for Q.
4. In double integrals, let $dudv$ be written as $\mathbf{du}$; thus $\mathbf{du}_p = du_p dv_p$ and $\mathbf{du}_q = du_q dv_q$. Also, let

$$\int_0^1\int_0^1 \text{ be written as } \iint_0^1$$

With this nomenclature, Eq. (A.26) can be expressed parametrically by

$$F_{p-q} = \iint_0^1\iint_0^1 \frac{H(\mathbf{u}_p,\mathbf{u}_q)}{A_p\pi\,|\mathbf{r}_p(\mathbf{u}_p)-\mathbf{r}_q(\mathbf{u}_q)|^2} J_p(\mathbf{u}_p)J_p(\mathbf{u}_q)\mathbf{du}_p\mathbf{du}_q \tag{A.27}$$

where

$$H(\mathbf{u}_p,\mathbf{u}_q) = \operatorname{poso}(\hat{\mathbf{n}}_p(\mathbf{u}_p)\cdot[-\hat{\mathbf{e}}((\mathbf{u}_p,\mathbf{u}_q)])\operatorname{poso}(\hat{\mathbf{n}}_q(\mathbf{u}_q)\cdot\hat{\mathbf{e}}(\mathbf{u}_p,\mathbf{u}_q)) \tag{A.28}$$

Figure A.2 shows a sample calculation of F_{p-q} for the case where surface P is a 2×1 rectangle in the vertical plane $y = -1$, having vertices $(1, -1, 0)$,

$(-1,-1,0)$, $(1,-1,1)$, $(-1,-1,1)$, and Q is a circle of radius 1 in the x-y plane, centered at the origin.

If evaluating the relevant quadruple integrals is taking an inordinate amount of time, as it sometimes does, one can use Monte Carlo integration. Monte Carlo integration is well suited for multiple integrals. Addendum 1 in Fig. A.2 shows a scheme for evaluating the quadruple integral using Monte Carlo integration. For Monte Carlo integration, the accuracy depends on the number N_R of random numbers used; the accuracy achieved generally varies as $1/\sqrt{N_R}$. Experience has shown that for the quadruple integrals arising in form factor evaluations, answers are typically within about 15% of the true one when using $N_R = 1000$, within about 4% when using $N_R = 10,000$, and within about 1% when using $N_R = 100,000$. The latter is often accurate enough for most engineering work, and the Monte Carlo quadruple integration using $N_R = 100,000$ takes only a fraction of the time taken by the Mathcad integrator.

A.3.5 Gaseous Form-Factor Evaluation

The gaseous form-factor function can be shown to be given by

$$G_{p-q}(a) = \iint_0^1 \iint_0^1 \frac{e^{-a|\mathbf{r}_p(\mathbf{u}_p)-\mathbf{r}_q(\mathbf{u}_q)|}H(\mathbf{u}_p,\mathbf{u}_q)}{A_p\pi\left|\mathbf{r}_p(\mathbf{u}_p)-\mathbf{r}_q(\mathbf{u}_q)\right|^2} J_p(\mathbf{u}_p)J_p(\mathbf{u}_q)\mathbf{du}_p\mathbf{du}_q \quad \text{(A.29)}$$

where $H(\mathbf{u}_p, \mathbf{u}_q)$ is as defined by Eq. (A.28). The Addendum 2 of Fig. A.2 shows a sample calculation of a gaseous form-factor function, using Monte Carlo integration. Here, surfaces P and Q are the same surfaces as the sample calculation in the previous section, and a has been set equal to 0.5.

A.4 Representing an Entire Enclosure

Sometimes we seek a parametric representation of the entire surface of an enclosed region, that is, of an enclosure. The interior surface of a sphere would be one such enclosure, but more often the enclosure is not a single geometric shape like a sphere, but is made up of several geometric shapes. We may know the parametric representation for each shape, but we seek the representation for the complete enclosure.

For example we may wish a parametric representation for the interior surface of a closed hemisphere. The hemisphere is made of two geometric parts: the half of a sphere and the circular base. We can use Entry 9 of the catalog to represent the first and Entry 2 to represent the second. But now we want a parametric representation for the entire surface. We can do this making the $\mathbf{r}(u,v)$ function for the entire enclosure a piecewise function. Let $\mathbf{r} = \mathbf{r}_1(u_1, v_1)$ and $\mathbf{r} = \mathbf{r}_2(u_2, v_2)$ be the representations of surfaces 1 and 2 respectively. Then we can define the $\mathbf{r}(u,v)$ function for the entire surface by

$$\begin{aligned} \mathbf{r}(u,v) &= \mathbf{r}_1(u,v) \qquad \text{for} \quad 0 \le u \le 1 \\ \text{and} \quad \mathbf{r}(u,v) &= \mathbf{r}_2(u-1,v) \qquad \text{for} \quad 1 \le u \le 2 \end{aligned} \quad \text{(A.30)}$$

We make the domain for the enclosure be $0 \leq u \leq 2$; $0 \leq v \leq 1$, so the enclosure's representation is not normalized. If we think about what happens when u and v cover their entire domains, we see that indeed the entire enclosure is accounted for. Surface 1 is accounted for when u and v extend over $0 \leq v \leq 1$ and $0 \leq u \leq 1$, and surface 2 when they extend over $0 \leq v \leq 1$ and $1 \leq u \leq 2$.

This example can be readily generalized to apply to enclosures with any number of geometric shapes or parts. If there are Ω parts represented individually by $\mathbf{r} = \mathbf{r}_1(u_1, v_1)$, $\mathbf{r} = \mathbf{r}_2(u_2, v_2)$...and $\mathbf{r} = \mathbf{r}_\Omega(u_\Omega, v_\Omega)$, respectively, then the enclosure would be represented by

$$\mathbf{r} = \mathbf{r}(u, v) \qquad 0 \leq v \leq 1; \quad 1 \leq u \leq \Omega \tag{A.31}$$

where $\mathbf{r}(u, v)$ is defined by

$$\begin{aligned}
\mathbf{r}(u,v) &= \mathbf{r}_1(u,v) && \text{for} \quad 0 \leq u \leq 1 \\
\mathbf{r}(u,v) &= \mathbf{r}_2(u-1,v) && \text{for} \quad 1 \leq u \leq 2 \\
\mathbf{r}(u,v) &= \mathbf{r}_3(u-2,v) && \text{for} \quad 2 \leq u \leq 3 \\
& \quad ... \\
\mathbf{r}(u,v) &= \mathbf{r}_\Omega\,(u-(\Omega-1),v) && \text{for} \quad \Omega-1 \leq u \leq \Omega
\end{aligned} \tag{A.32}$$

A similar set of equations would apply for the unit normal $\hat{\mathbf{n}}(u, v)$ to the surface and the area factor $J(u, v)$ for the enclosure. Thus, for example, $\hat{\mathbf{n}}(u, v) = \mathbf{n}_3(u-2, v)$ and $J(u, v) = J_3(u-2, v)$ whenever $2 \leq u \leq 3$, it being understood that $\mathbf{n}_3(u, v)$.and $J_3(u, v)$ are, respectively, the unit normal and area factor for part 3.

$i := (1\ \ 0\ \ 0)^T \qquad j := (0\ \ 1\ \ 0)^T \qquad k := (0\ \ 0\ \ 1)^T$

Solid Angle Calculator:

The solid angle is subtended from point rp=(xp,yp,zp)

$rp := (2\ \ 1\ \ 1)^T$

The shape subtended is a:

Circle: Centred at r1, radius R, normal N

$N := (1\ \ 1\ \ 1)^T \qquad n(u,v) := \dfrac{N}{|N|} \qquad nn := \dfrac{N}{|N|} \qquad R := 2 \qquad r1 := (1\ \ 2\ \ 3)^T$

$ij := \mathrm{if}(|nn \times i| = 0, j, i) \qquad m := \dfrac{ij \times nn}{|ij \times nn|}$

$r(u,v) := r1 + R \cdot u \cdot (m \cdot \cos(2 \cdot \pi \cdot v) + m \times nn \cdot \sin(2\pi \cdot v)) \qquad J(u,v) := 2 \cdot \pi \cdot u \cdot R^2 \qquad As := \pi \cdot R^2$

Setting up Integration

$e(u,v) := \dfrac{r(u,v) - rp}{|r(u,v) - rp|} \qquad poso(x) := \dfrac{x + |x|}{2}$

$$\omega := \int_0^1 \int_0^1 \frac{poso(n(u,v) \cdot e(u,v)) \cdot J(u,v)}{(|r(u,v) - rp|)^2}\, du\, dv \qquad \omega = 1.429396$$

Addendum 1: Extension to Point Form Factor Calculation

The Point Form Factor is from an elemental area d1 at the same point rp=(xp,yp,zp) and with (inward) unit normal in direction n1

$n1 := (0\ \ 0\ \ -1)^T$

$$Fd1 := \int_0^1 \int_0^1 \frac{poso(n1 \cdot -e(u,v)) \cdot poso(n(u,v) \cdot e(u,v)) \cdot J(u,v)}{\pi(|r(u,v) - rp|)^2}\, du\, dv \qquad Fd1 = 0.336747$$

Addendum 2: Extension to Gaseous Point Form Factor Calculation

The same geometric conditions apply as for Addendum 1

$$Gd1(a) := \int_0^1 \int_0^1 \frac{\exp(-a \cdot |r(u,v) - rp|) \cdot poso(-k \cdot -e(u,v)) \cdot poso(n(u,v) \cdot e(u,v)) \cdot J(u,v)}{\pi(|r(u,v) - rp|)^2}\, du\, dv$$

$Gd1(.5) = 0.134912$

Figure A.1: Mathcad worksheet illustrating the determination of a solid angle, a point form factor, and a gaseous point form factor using Mathcad. The example considers the surface to be a circle in 3-space.

Preliminary definitions :

$i := (1\ 0\ 0)^T \quad j := (0\ 1\ 0)^T \quad k := (0\ 0\ 1)^T \quad \text{poso}(x) := \frac{x + |x|}{2}$

Form Factor, Rectangle P to Circle Q

Rectangle P has vertices having position vectors r1p, r2p, r3p

$r1p := (1\ -1\ 0)^T \quad r2p := (-1\ -1\ 0)^T \quad r3p := (1\ -1\ 1)^T$

$rp(u,v) := r1p + (r2p - r1p)\cdot u + (r3p - r1p)\cdot v \qquad Asp := |r1p - r2p|\cdot|r1p - r3p|$

$np(u,v) := \frac{-(r1p - r2p) \times (r1p - r3p)}{|(r1p - r2p) \times (r1p - r3p)|} \qquad Jp(u,v) := |r1p - r2p|\cdot|r1p - r3p|$

Circle Q has Radius Rq, Center, rcq, normal Nq:

$Nq := (0\ 0\ -1)^T \quad nq(u,v) := \frac{Nq}{|Nq|} \quad nnq := \frac{Nq}{|Nq|} \quad Rq := 1 \quad r1q := (0\ 0\ 0)^T$

$ijq := \text{if}(|nnq \times i| = 0, j, i) \qquad mq := \frac{ijq \times nnq}{|ijq \times nnq|}$

$rq(u,v) := r1q + Rq\cdot u\cdot(mq\cdot\cos(2\cdot\pi\cdot v) + mq \times nnq\cdot\sin(2\pi\cdot v)) \quad Jq(u,v) := 2\cdot\pi\cdot u\cdot Rq^2 \quad Asq := \pi\cdot Rq^2$

Setting up Integrand:

$rpq(up,vp,uq,vq) := (rp(up,vp) - rq(uq,vq)) \qquad epq(up,vp,uq,vq) := \frac{rpq(up,vp,uq,vq)}{|rpq(up,vp,uq,vq)|}$

$$I(up,vp,uq,vq) := \frac{\text{poso}(np(up,vp)\cdot epq(up,vp,uq,vq))\cdot\text{poso}(nq(uq,vq)\cdot -epq(up,vp,uq,vq))\cdot Jp(up,vp)\cdot Jq(uq,vq)}{\pi\cdot(|rpq(up,vp,uq,vq)|)^2\cdot Asp}$$

Integrating:

$$Fpq := \int_0^1\int_0^1\int_0^1\int_0^1 I(up,vp,uq,vq)\,dup\,dvp\,duq\,dvq \qquad Fpq = 0.2177 \quad Fqp := \frac{Asq}{Asp}\cdot Fpq \qquad Fqp = 0.3419$$

Addendium 1: Using Monte Carlo (MC) Intergration

$$FpqMC(NR) := \frac{\sum_{i=0}^{NR} I(\text{runif}(4+NR,0,1)_i, \text{runif}(4+NR,0,1)_{i+1}, \text{runif}(4+NR,0,1)_{i+2}, \text{runif}(4+NR,0,1)_{i+3})}{(NR+1)}$$

$FpqMC(1000) = 0.2189 \qquad FpqMC(10000) = 0.2093 \qquad FpqMC(100000) = 0.2159$

$FqpMC := \frac{Asq}{Asp}\cdot 0.2188 \qquad FqpMC = 0.3437$

Addendium 2: Gaseous Form Factor Evaluation

$I1(up,vp,uq,vq,a) := I(up,vp,uq,vq)\exp(-a\cdot|rpq(up,vp,uq,vq)|) \qquad NR := 100000$

$$Gpq(a) := \frac{\sum_{i=0}^{NR} I1(\text{runif}(4+NR,0,1)_i, \text{runif}(4+NR,0,1)_{i+1}, \text{runif}(4+NR,0,1)_{i+2}, \text{runif}(4+NR,0,1)_{i+3}, a)}{(NR+1)}$$

$Gpq(.5) = 0.1352$

Figure A.2: Mathcad worksheet illustrating the determination of form factors and gaseous form factor using numerical integration and the parametric representation of surfaces