APPENDIX A

CURVILINEAR COORDINATES

Let

$$\mathbf{R} = x_1 g_1 + x_2 g_2 + x_3 g_3 \equiv x_\alpha g_\alpha \tag{A.1}$$

be the radius vector of a point in physical space (see Figure A.1). In this equation the x_i are rectangular Cartesian coordinates of the point, and $\mathbf{g}_i$ are coordinate unit vectors. In what follows, the repetition of a *Greek* index implies summation with respect to that index from 1 to 3.

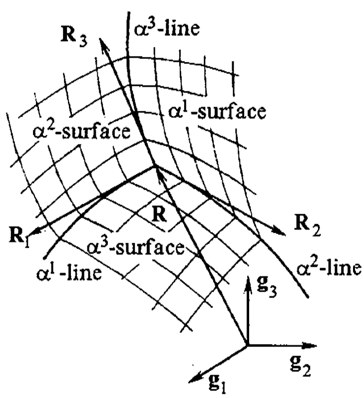

FIGURE A.1

Curvilinear coordinates $\alpha^1, \alpha^2, \alpha^3$ are defined by the relations

$$x_i = x_i(\alpha^1, \alpha^2, \alpha^3) \qquad (i = 1, 2, 3). \tag{A.2}$$

The equation $\alpha^i = \alpha^i_0 =$ constant determines the ith *coordinate surface*, and the relations $\alpha^i = \alpha^i_0, \alpha^j = \alpha^j_0$ define the kth *coordinate line* ($k \neq i \neq j \neq k$). The vectors

$$\mathbf{R}_i \equiv \frac{\partial \mathbf{R}}{\partial \alpha^i} = \frac{\partial x_1}{\partial \alpha^i}\mathbf{g}_1 + \frac{\partial x_2}{\partial \alpha^i}\mathbf{g}_2 + \frac{\partial x_3}{\partial \alpha^i}\mathbf{g}_3 \equiv \frac{\partial x_\alpha}{\partial \alpha^i}\mathbf{g}_\alpha \tag{A.3}$$

are tangent (see Figure A.1) to coordinate α^i-lines and are of lengths

$$A_i \equiv |\mathbf{R}_i| = \sqrt{\left(\frac{\partial x_1}{\partial \alpha^i}\right)^2 + \left(\frac{\partial x_2}{\partial \alpha^i}\right)^2 + \left(\frac{\partial x_3}{\partial \alpha^i}\right)^2} = \sqrt{\frac{\partial x_\alpha}{\partial \alpha^i}\frac{\partial x_\alpha}{\partial \alpha^i}}. \tag{A.4}$$

These are the so-called *Lamé parameters.* It is obvious that the quantities

$$\mathbf{e}_i = \frac{\mathbf{R}_i}{A_i} = \frac{1}{A_i}\frac{\partial x_\alpha}{\partial \alpha^i}\mathbf{g}_\alpha \qquad (i = 1, 2, 3) \tag{A.5}$$

are *unit coordinate vectors* that, in general, are not mutually orthogonal.

It follows from the system of equations (A.5) that

$$\mathbf{g}_k = (\mu_k/\mu)\mathbf{e}_\beta\,, \tag{A.6}$$

where

$$\begin{aligned}
A_2A_3\mu_{11} &= \frac{\partial x_2}{\partial \alpha^2}\frac{\partial x_3}{\partial \alpha^3} - \frac{\partial x_2}{\partial \alpha^3}\frac{\partial x_3}{\partial \alpha^2}, & A_3A_1\mu_{12} &= \frac{\partial x_2}{\partial \alpha^3}\frac{\partial x_3}{\partial \alpha^1} - \frac{\partial x_2}{\partial \alpha^1}\frac{\partial x_3}{\partial \alpha^3},\\
A_1A_2\mu_{13} &= \frac{\partial x_2}{\partial \alpha^1}\frac{\partial x_3}{\partial \alpha^2} - \frac{\partial x_2}{\partial \alpha^2}\frac{\partial x_3}{\partial \alpha^1}, & A_2A_3\mu_{21} &= \frac{\partial x_3}{\partial \alpha^2}\frac{\partial x_1}{\partial \alpha^3} - \frac{\partial x_3}{\partial \alpha^3}\frac{\partial x_1}{\partial \alpha^2},\\
A_3A_1\mu_{22} &= \frac{\partial x_3}{\partial \alpha^3}\frac{\partial x_1}{\partial \alpha^1} - \frac{\partial x_3}{\partial \alpha^1}\frac{\partial x_1}{\partial \alpha^3}, & A_1A_2\mu_{23} &= \frac{\partial x_3}{\partial \alpha^1}\frac{\partial x_1}{\partial \alpha^2} - \frac{\partial x_3}{\partial \alpha^2}\frac{\partial x_1}{\partial \alpha^1},\\
A_2A_3\mu_{31} &= \frac{\partial x_1}{\partial \alpha^2}\frac{\partial x_2}{\partial \alpha^3} - \frac{\partial x_1}{\partial \alpha^3}\frac{\partial x_2}{\partial \alpha^2}, & A_3A_1\mu_{32} &= \frac{\partial x_1}{\partial \alpha^3}\frac{\partial x_2}{\partial \alpha^1} - \frac{\partial x_1}{\partial \alpha^1}\frac{\partial x_2}{\partial \alpha^3},\\
A_1A_2\mu_{33} &= \frac{\partial x_1}{\partial \alpha^1}\frac{\partial x_2}{\partial \alpha^2} - \frac{\partial x_1}{\partial \alpha^2}\frac{\partial x_2}{\partial \alpha^1}, & \frac{\mu}{A_1A_2A_3} &= \begin{vmatrix} \frac{\partial x_1}{\partial \alpha^1} & \frac{\partial x_2}{\partial \alpha^1} & \frac{\partial x_3}{\partial \alpha^1}\\ \frac{\partial x_1}{\partial \alpha^2} & \frac{\partial x_2}{\partial \alpha^2} & \frac{\partial x_3}{\partial \alpha^2}\\ \frac{\partial x_1}{\partial \alpha^3} & \frac{\partial x_2}{\partial \alpha^3} & \frac{\partial x_3}{\partial \alpha^3} \end{vmatrix}.
\end{aligned} \tag{A.7}$$

Apart from the *basic coordinate* vectors

$$\mathbf{R}_1 = A_1\mathbf{e}_1\,, \qquad \mathbf{R}_2 = A_2\mathbf{e}_2\,, \qquad \mathbf{R}_3 = A_3\mathbf{e}_3\,, \tag{A.8$_{1-3}$}$$

the *reciprocal coordinate* vectors

$$\mathbf{R}^1 = \frac{\mathbf{R}_2 \times \mathbf{R}_3}{\mathbf{R}_1 \cdot (\mathbf{R}_2 \times \mathbf{R}_3)}, \qquad \mathbf{R}^2 = \frac{\mathbf{R}_3 \times \mathbf{R}_1}{\mathbf{R}_1 \cdot (\mathbf{R}_2 \times \mathbf{R}_3)}, \qquad \mathbf{R}^3 = \frac{\mathbf{R}_1 \times \mathbf{R}_2}{\mathbf{R}_1 \cdot (\mathbf{R}_2 \times \mathbf{R}_3)} \tag{A.8$_{4-6}$}$$

are also used; they satisfy the basic *reciprocity conditions*

$$\mathbf{R}_i \cdot \mathbf{R}^j = \delta_i^j = \begin{cases} 1, & i = j,\\ 0, & i \neq j. \end{cases} \tag{A.9}$$

It is easily verified that

$$A_j \mathbf{R}^j = \frac{\mu_{\alpha j}}{\mu} \mathbf{g}_\alpha = A_j \frac{\partial \alpha^j}{\partial x_\beta} \mathbf{g}_\beta. \tag{A.10}$$

The quantities

$$g_{ij} = \mathbf{R}_i \cdot \mathbf{R}_j = g_{ji}, \qquad g^{ij} = \mathbf{R}^i \cdot \mathbf{R}^j = g^{ji} \\ (A_i = \sqrt{g_{ii}}) \tag{A.11}$$

connect the basic and reciprocal vectors by the relations

$$\mathbf{R}_i = g_{i\alpha} \mathbf{R}^\alpha, \qquad \mathbf{R}^j = g^{\beta j} \mathbf{R}_\beta. \tag{A.12}$$

Vectors and tensors are represented by the expansions in coordinate bases:

$$\begin{aligned} \mathbf{u} &= u_\alpha \mathbf{R}^\alpha = u^\alpha \mathbf{R}_\alpha, \\ \mathbf{T} &= t^{\alpha\beta} \mathbf{R}_\alpha \mathbf{R}_\beta = t^{\alpha}_{.\beta} \mathbf{R}_\alpha \mathbf{R}^\beta = t_{\alpha}^{.\beta} \mathbf{R}^\alpha \mathbf{R}_\beta = t_{\alpha\beta} \mathbf{R}^\alpha \mathbf{R}^\beta, \\ \mathbf{W} &= w^{\alpha\beta\dots}_{\dots\mu\nu} \mathbf{R}_\alpha \mathbf{R}_\beta \dots \mathbf{R}^\mu \mathbf{R}^\nu = \dots . \end{aligned} \tag{A.13}$$

The quantities $\mathbf{R}_i\mathbf{R}_j$, $\mathbf{R}_i\mathbf{R}^j$, $\mathbf{R}^i\mathbf{R}_j$, $\mathbf{R}^i\mathbf{R}^j$ are called *coordinate dyads*, and $\mathbf{R}_i\mathbf{R}_j \dots \mathbf{R}^k\mathbf{R}^l$ are called *coordinate polyads*. In accord with formulas (A.9) and (A.11), these quantities possess the following properties:

$$\begin{aligned} &\mathbf{R}_i\mathbf{R}^j \cdot \mathbf{R}^k = g^{jk}\mathbf{R}_i, \qquad \mathbf{R}^k \cdot \mathbf{R}_i\mathbf{R}^j = \delta_i^k \mathbf{R}^j, \\ &\mathbf{R}_i\mathbf{R}_j \cdot \mathbf{R}^k\mathbf{R}_l = \delta_j^k \mathbf{R}_i\mathbf{R}_l, \qquad \mathbf{R}_i\mathbf{R}^j \cdot \mathbf{R}^k\mathbf{R}^l = g^{jk}\mathbf{R}_i\mathbf{R}^l, \dots, \\ &\dots, \mathbf{R}_i\mathbf{R}_j \dots \mathbf{R}^l\mathbf{R}_m \cdot \mathbf{R}_j\mathbf{R}_t = g_{mj}\mathbf{R}_i\mathbf{R}_j \dots \mathbf{R}^l\mathbf{R}_t, \dots, \end{aligned} \tag{A.14}$$

The expansion coefficients in (A.13) with superscripts are called *contravariant*, those with subscripts *covariant*, and those with superscripts and subscripts are called *mixed* components. The points in mixed components determine the sequence of indices. For symmetric tensors, there is no point in indicating the arrangement of indices, and so the points are dropped.

With the help of the relations (A.14) the connection between different components of one and the same tensor (vector) is established:

$$\begin{aligned} u_i &= g_{i\alpha} u^\alpha, \qquad u^j = g^{j\beta} u_\beta, \\ t^{ij} &= g^{i\alpha} t_{\alpha}^{.j} = g^{j\beta} t^{i}_{.\beta} = g^{i\alpha} g^{j\beta} t_{\alpha\beta}, \\ t_{ij} &= g_{i\alpha} t^{\alpha}_{.j} = g_{j\beta} t_{i}^{.\beta} = g_{i\alpha} g_{j\beta} t^{\alpha\beta}. \end{aligned} \tag{A.15}$$

Using scalar multiplication by $\mathbf{R}^j$ of the first of the relations (A.12) gives

$$g_{i\alpha} g^{\alpha j} = \delta_i^j. \tag{A.16}$$

The tensor

$$\mathbf{1} = g_{\alpha\beta} \mathbf{R}^\alpha \mathbf{R}^\beta = \mathbf{R}^\alpha \mathbf{R}_\alpha = \mathbf{R}_\alpha \mathbf{R}^\alpha = g^{\alpha\beta} \mathbf{R}_\alpha \mathbf{R}_\beta \mathbf{G} \tag{A.17}$$

is called *unitary*, since by (A.14) for any tensor we have

$$\mathbf{T} \cdot \mathbf{1} = \mathbf{1} \cdot \mathbf{T} = \mathbf{T}. \tag{A.18}$$

This tensor is also called the *metric tensor*, because, knowing its components, we can perform various metric operations. Thus the length of an element of arc is determined by the formula

$$ds^2 = g_{\alpha\beta}\, d\alpha^\alpha\, d\alpha^\beta. \tag{A.19}$$

In particular, for an element of arc belonging to the ith coordinate line we have

$$ds_i = \sqrt{g_{ii}}\, d\alpha^i = A_i\, d\alpha^i. \tag{A.20}$$

The angle between the elements of tangents to the intersecting curves $d\mathbf{R}(d\alpha^1, d\alpha^2, d\alpha^3)$ and $\delta\mathbf{R}(\delta\alpha^1, \delta\alpha^2, \delta\alpha^3)$ is determined by

$$\cos\chi = \frac{\delta\mathbf{R} \cdot d\mathbf{R}}{|\delta\mathbf{R}| \cdot |d\mathbf{R}|} = \frac{g_{\alpha\beta}\, \delta\alpha^\alpha\, d\alpha^\beta}{\sqrt{g_{\alpha\beta}\, d\alpha^\alpha\, d\alpha^\beta} \sqrt{g_{\alpha\beta}\, \delta\alpha^\alpha\, \delta\alpha^\beta}}. \tag{A.21}$$

In particular, the angle between the ith and jth coordinate lines is given by

$$\cos\chi^{(k)} = \frac{g_{ij}}{\sqrt{g_{ii}}\sqrt{g_{jj}}} \qquad (k \neq i \neq j \neq k). \tag{A.22}$$

The *discriminant* tensor is determined by its covariant and contravariant components

$$\varepsilon_{ijk} = \mathbf{R}_i \cdot (\mathbf{R}_j \times \mathbf{R}_k), \qquad \varepsilon^{ijk} = \mathbf{R}^i \cdot (\mathbf{R}^j \times \mathbf{R}^k). \tag{A.23}$$

Observe that only the components

$$\begin{aligned} &\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = -\varepsilon_{132} = -\varepsilon_{321} = -\varepsilon_{213} = \sqrt{g}, \\ &\varepsilon^{123} = \varepsilon^{231} = \varepsilon^{312} = -\varepsilon^{132} = -\varepsilon^{321} = -\varepsilon^{213} = 1/\sqrt{g}. \end{aligned} \tag{A.24}$$

are different from zero (here $g = |g_{ij}|$). Further, we have

$$\mathbf{R}_i \times \mathbf{R}_j = \varepsilon_{ij\alpha}\mathbf{R}^\alpha, \qquad \mathbf{R}^i \times \mathbf{R}^j = \varepsilon^{ij\alpha}\mathbf{R}_\alpha. \tag{A.25}$$

The area of the ith coordinate surface element is determined by

$$dS_i = \sqrt{gg^{ii}}\, d\alpha^j\, d\alpha^k \qquad (i \neq j \neq k \neq i). \tag{A.26}$$

A volume element dV is equal to

$$dV = \sqrt{g}\, d\alpha^1\, d\alpha^2\, d\alpha^3. \tag{A.27}$$

Finally, we have

$$\mathbf{n}\, dS_n = \frac{\mathbf{R}^\alpha}{\sqrt{g^{\alpha\alpha}}} dS_\alpha, \tag{A.28}$$

$$n_i = \frac{1}{\sqrt{g^{ii}}} \frac{dS_i}{dS_n}. \tag{A.29}$$

where $\mathbf{n}$ is the unit normal vector to the oblique face (see Figure A.2).

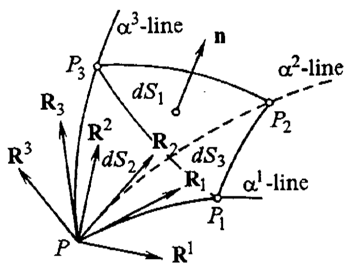

FIGURE A.2

For differentiation of unit coordinate vectors, the following formulas are used:

$$\frac{\partial \mathbf{R}_j}{\partial \alpha^i} = G_{ij}^\alpha \mathbf{R}_\alpha, \tag{A.30}$$

$$\frac{\partial \mathbf{R}^k}{\partial \alpha^j} = -G_{j\alpha}^k \mathbf{R}^\alpha, \tag{A.31}$$

where

$$G_{ij}^k = \frac{1}{2}\left(\frac{\partial g_{i\beta}}{\partial \alpha^j} + \frac{\partial g_{j\beta}}{\partial \alpha^i} - \frac{\partial g_{ij}}{\partial \alpha^\beta}\right) g^{\beta k} \tag{A.32}$$

are the *Christoffel symbols* of the first kind.

In the study of general problems of the theory of elasticity covariant derivatives of components of vectors and tensors are often used. Thus the quantities

$$\nabla_i u_j = \frac{\partial u_j}{\partial \alpha^i} - G_{ij}^{\gamma} u_\gamma, \qquad \nabla_i u^j = \frac{\partial u^j}{\partial \alpha^i} + G_{\gamma i}^{j} u^\gamma; \tag{A.33}$$

$$\begin{aligned} \nabla_k t^i_{\cdot j} &= \frac{\partial t^i_{\cdot j}}{\partial \alpha^k} + G^i_{\gamma k} t^\gamma_{\cdot j} - G^\gamma_{jk} t^i_{\cdot \gamma}, \\ \nabla_k t_i^{\cdot j} &= \frac{\partial t_i^{\cdot j}}{\partial \alpha^k} - G^\gamma_{ik} t^j_\gamma + G^{\cdot j}_{\gamma k} t_i^{\cdot j}, \\ \nabla_k t^{ij} &= \frac{\partial t^{ij}}{\partial \alpha^k} + G^i_{\gamma k} t^{\gamma j} + G^j_{\gamma k} t^{i\gamma}, \\ \nabla_k t_{ij} &= \frac{\partial t_{ij}}{\partial \alpha^k} - G^\gamma_{ik} t_{\gamma j} - G^\gamma_{jk} t_{i\gamma} \end{aligned} \tag{A.34}$$

are *covariant derivatives* of the components of an arbitrary vector and of a tensor of rank two. (The *rank* of a tensor is equal to the number of indices of the components.)

Covariant differentiation adds a covariant (lower) index to the components of the tensor, making them components of a tensor whose rank is larger by one; this tensor is the so-called *tensorial gradient* [see formula (A.13)]:

$$\nabla \mathbf{T} = (\nabla_\gamma t_{\alpha\beta}) \mathbf{R}^\gamma \mathbf{R}^\alpha \mathbf{R}^\beta = (\nabla_\gamma t^{\alpha\beta}) \mathbf{R}^\gamma \mathbf{R}_\alpha \mathbf{R}_\beta = \cdots. \tag{A.35}$$

The covariant derivative has a number of remarkable properties:

$$\begin{aligned} &\nabla_i \mathbf{R}_j = 0, \qquad \nabla_i \mathbf{R}^j = 0, \\ &\nabla_k g_{ij} = 0, \qquad \nabla_k g^{ij} = 0, \qquad \nabla_k \delta_i^j = 0, \\ &\nabla_k \varepsilon_{ijl} = 0, \qquad \nabla_k \varepsilon^{ijl} = 0. \end{aligned} \tag{A.36}$$

Thus, in covariant differentiation, coordinate vectors, components of the metric and discriminant tensors behave like constants.

Observe that the operation of covariant differentiation was defined for the *components* of vectors and tensors. The tensors (vectors) themselves are invariant (with respect to the choice of the coordinate system) quantities (having no indices). The covariant derivative for them coincides with the partial derivative. Therefore

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial \alpha^i} &\equiv \nabla_i \mathbf{u} = \nabla_i (u_\nu \mathbf{R}^\nu) = (\nabla_i u_\nu) \mathbf{R}^\nu \\ &= \nabla_i (u^\nu \mathbf{R}_\nu) = (\nabla_i u^\nu) \mathbf{R}_\nu, \\ \frac{\partial \mathbf{T}}{\partial \alpha^i} &\equiv \nabla_i \mathbf{T} = \nabla_i (t_{\alpha\beta} \mathbf{R}^\alpha \mathbf{R}^\beta) = (\nabla_i t_{\alpha\beta}) \mathbf{R}^\alpha \mathbf{R}^\beta = \cdots. \end{aligned} \tag{A.37}$$

The order of covariant differentiation is immaterial (in Euclidean space).

The following useful relations are valid:

$$\nabla_k(\sqrt{g}) = 0, \quad \nabla_\gamma u^\gamma = \frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}\,u^\gamma}{\partial\alpha^\gamma}, \quad \nabla_\gamma(\sqrt{g}u^\gamma) = \frac{\partial\sqrt{g}\,u^\gamma}{\partial\alpha^\gamma}\,; \tag{A.38}$$

$$\nabla_\gamma t^{\gamma j} = \frac{1}{\sqrt{g}}\left(\frac{\partial\sqrt{g}\,t^{\gamma j}}{\partial\alpha^\gamma} + G^j_{\gamma\beta}\sqrt{g}\,t^{\gamma\beta}\right),$$
$$\nabla_\gamma(\sqrt{g}t^{\gamma j}) = \frac{\partial\sqrt{g}\,t^{\gamma j}}{\partial\alpha^\gamma} + G^j_{\gamma\beta}\sqrt{g}\,t^{\gamma\beta}\,; \tag{A.39}$$

$$g^{ij} = \frac{1}{g}\frac{\partial g}{\partial g_{ij}}. \tag{A.40}$$

In applications *orthogonal* coordinates are mostly used, in which case the unit vectors $\mathbf{e}_i$ (A.5) are unit base vectors, i.e.,

$$\begin{gathered}
\mathbf{e}_1\cdot\mathbf{e}_1 = \mathbf{e}_2\cdot\mathbf{e}_2 = \mathbf{e}_3\cdot\mathbf{e}_3 = 1, \qquad \mathbf{e}_1\cdot\mathbf{e}_2 = \mathbf{e}_2\cdot\mathbf{e}_3 = \mathbf{e}_3\cdot\mathbf{e}_1 = 0\,;\\
\mathbf{e}_1\times\mathbf{e}_2 = \mathbf{e}_3, \qquad \mathbf{e}_2\times\mathbf{e}_3 = \mathbf{e}_1, \qquad \mathbf{e}_3\times\mathbf{e}_1 = \mathbf{e}_2\,;\\
\mathbf{R}_i = A_i\mathbf{e}_i = \sqrt{a_{ii}}\,\mathbf{e}_i, \qquad \mathbf{R}^j = \mathbf{e}_j/A_j = \mathbf{e}_j/\sqrt{a_{jj}}\,;\\
g^{ii} = g_{ii}^{-1}, \qquad g = g_{11}g_{22}, \qquad g_{ij} = 0, \qquad g^{ij} = 0 \qquad (i \neq j).
\end{gathered} \tag{A.41}$$

The components of vectors and tensors in the representations

$$\begin{gathered}
\mathbf{u} = \mathbf{u}_{(\alpha)}\mathbf{e}_\alpha, \qquad \mathbf{T} = t_{(\alpha\beta)}\mathbf{e}_\alpha\mathbf{e}_\beta,\\
\mathbf{W} = w_{(\alpha\beta\ldots\mu\nu)}\mathbf{e}_\alpha\mathbf{e}_\beta\ldots\mathbf{e}_\mu\mathbf{e}_\nu
\end{gathered} \tag{A.42}$$

are called *physical*. In view of (A.41), a comparison of these expressions with the expansions (A.13) gives

$$\begin{aligned}
u_{(i)} &= u^i\sqrt{g_{ii}} = u_i/\sqrt{g_{ii}},\\
t_{(ij)} &= t_{ij}/\sqrt{g_{ii}g_{jj}} = t^{ij}\sqrt{g_{ii}g_{jj}}\\
&= t^i_{\cdot j}\sqrt{g_{ii}}/\sqrt{g_{jj}} = t_i^{\cdot j}\sqrt{g_{jj}}/\sqrt{g_{ii}},\\
w_{(ij\ldots kl)} &= w^{ij\ldots}_{\ldots kl}\sqrt{g_{ii}g_{jj}\ldots g_{kk}^{-1}g_{ll}^{-1}}.
\end{aligned} \tag{A.43}$$

Apart from the original coordinate system, we shall consider a new ("primed") coordinate system, which is also orthogonal. We have

$$\mathbf{e}'_j = \mathbf{e}_\alpha q_{\alpha j}, \qquad \mathbf{e}_k = q_{k\beta}\mathbf{e}'_\beta, \tag{A.44}$$

where the q_{ij} are the cosines of angles of rotation, which are connected with the components of the rotation vector $\boldsymbol{\omega} = \omega_\alpha \mathbf{e}_\alpha$ by the relation

$$\begin{Vmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{Vmatrix} = \cos\omega \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix} + \frac{\sin\omega}{\omega} \begin{Vmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{Vmatrix}$$
$$+ \frac{1-\cos\omega}{\omega^2} \begin{Vmatrix} \omega_1^2 & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & \omega_2^2 & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & \omega_3^2 \end{Vmatrix}$$
$$\left(\omega = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}\right). \tag{A.45}$$

The inverse relations hold:

$$\cos\omega = \tfrac{1}{2}(q_{11} + q_{22} + q_{33} - 1),$$
$$\frac{\omega_1}{\omega} = \frac{q_{32} - q_{23}}{2\sin\omega}, \qquad \frac{\omega_2}{\omega} = \frac{q_{13} - q_{31}}{2\sin\omega}, \qquad \frac{\omega_3}{\omega} = \frac{q_{21} - q_{12}}{2\sin\omega}. \tag{A.46}$$

It follows from the relations (A.44) and (A.42) that the physical components of vectors and tensors in the old and the new coordinate system are related by

$$a'_{(j)} = a_{(\alpha)} q_{\alpha j}, \qquad t'_{(ij)} = t_{(\alpha\beta)} q_{\alpha i} q_{\beta j},$$
$$w'_{(ij\ldots kl)} = w_{(\alpha\beta\ldots\gamma\delta)} q_{\alpha i} q_{\beta j} \cdots q_{\gamma k} q_{\delta l}. \tag{A.47}$$

If the passage from one orthogonal coordinate system to another one is associated with the transformation of coordinates

$$\alpha'^i = \alpha'^i(\alpha^1, \alpha^2, \alpha^3) \qquad (i = 1, 2, 3),$$

then

$$q_{ik} = \sqrt{g'_{ii}/g_{kk}} \frac{\partial \alpha'^i}{\partial \alpha^k}. \tag{A.48}$$

Suppose that the transformation of coordinates reduces to reflection in the mth coordinate plane (tangent to the coordinate surface). In this case

$$a'_{(j)} = a_{(j)}(-1)^p, \qquad t'_{(ij)} = t_{(ij)}(-1)^p,$$
$$w'_{(ij\ldots kl)} = w_{(ij\ldots kl)}(-1)^p, \tag{A.49}$$

where p is the number of indices of the component, which is equal to m.

In conclusion, let us cite the relations

$$\mathbf{T} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{T}^*, \qquad \mathbf{T}^* \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{T} \tag{A.50}$$

that connect arbitrary quantities—a vector $\mathbf{a}$, a tensor $\mathbf{T}$, and the conjugate tensor $\mathbf{T}^*$.