APPENDIX ONE

ANALYTICAL SOLUTIONS OF THE TEMPERATURE FIELDS IN LIQUID LAMINAR FLOWS

This chapter contains the analytical solutions of the temperature fields in liquids laminar flows. In the second, third and fourth chapters these results are used in the laminar flow methods and measuring devices.

§Ap.1.1 Application of the Green Function Method to Determine the Temperature Fields in Laminar Flows in Tubes

One of the important engineering problems in thermophysics is the calculation of temperature fields in steady laminar flow of liquids in tubes with appropriate boundary conditions and internal heat sources. Considering the analytical solutions of these problems in the form of Green functions [24, 37, 91-99], it is necessary to consider the following:

1. When the function of the internal heat source W depends only on the longitudinal z and transverse r coordinates, i.e. $W = W(r,z)$, it is enough to plot only once the Green functions of the corresponding boundary value problems with boundary conditions of the 1,2,3 type or mixed boundary conditions in order to obtain the solutions for all possible functions $W(r,z)$ if the functions of the initial distribution $T_b(r)$ and the functions $f(z)$, are preset on the tube boundaries.

2. When $W = W(r,z,T)$ is a non-linear temperature function, the solution involves a non-linear integral equation, which is obtained by the substitution of $W = W(r,z,T)$ in one of the formulas /Ap.1.4/, /Ap.1.8/, /Ap.1.12/. These formulas correspond to specific boundary conditions. If $W(r,z,T)$ satisfies the Lipshits conditions according to the third argument, then successive iterations [37] converge for the non-linear integral equations.

In many applied problems it is possible to obtain non-linear boundary problem solutions with high accuracy, if one presets as the zeroth iteration the solution of the linear boundary value problem or another approximate solution.

It is then possible to transform the problems with the non-linear boundary conditions to the problem with linear boundary conditions, but with a non-linear right part, W [37].

Let us consider the heat transfer problem of laminar flow of Newtonian and non-Newtonian liquids in tubes with either flat or round cross- sections with the following assumptions [34-36, 91-94]:

1. The pressure, flow and the heat transfer process are steady state.
2. The liquid is incompressible, its thermophysical properties are constant, i.e. they do not depend on temperature or pressure. We note that in the heat exchange calculations all liquids are considered as incompressible. Also, gases may be included, if their velocity is sufficiently less than the sound velocity (in practice the gas velocity must not exceed 0.3 of the sound velocity [100]).
3. The liquid flow is fully developed. The velocity profile doesn't vary along the tube length (a hydrodynamic entrance section, in which the velocity profile is fully developed, precedes the heat exchange section). The velocity profile and the liquid flow rate are preset. For flat, round, circular and rectangular tube cross sections, formulas for the velocity profile are given in [34-36].
4. At the inlet of the heat exchange section the liquid temperature distribution is known.
5. The type of boundary condition on the inner surface of the tube wall is known.
6. Internal heat sources are operating in the flow.
7. The heat flux change along the tube axis, due to liquid thermal conductivity, is small compared to the heat flux change, due to heat transfer by means of forced convection.

A problem considering the assumptions, close to those listed above, was first solved for Newtonian liquids by Graetz [40, 41] and later, independent of Graetz, Nusselt considered it a second time [44]. A somewhat different solution was obtained by Shumilov and Yablonsky [43]. Information about subsequent solutions are given in [34-36].

Considering assumptions 1-6 the energy equation for the case of the liquid laminar motion in the flat or cylindrical tube can be recorded in the form [34]:

$$\omega_z \frac{\partial T}{\partial z} = a\left(\frac{\partial^2 T}{\partial z^2} + \frac{1}{r^r}\cdot\frac{\partial}{\partial r}\left[r^r \frac{\partial T}{\partial r}\right]\right) + \frac{W}{c\rho}, \qquad z>0, \qquad r_1<r<r_2, \qquad /Ap.1.1/$$

where r, z are the radial and longitudinal coordinates; T - is the temperature, ω_z - is the flow velocity; a - is the liquid thermal diffusivity; c - is the specific heat capacity; ρ -is the density; r - is the shape coefficient (r=0 for the flat tube, r=1 for the round tube); W - is the function of the internal heat sources; r_1, r_2 - are the coordinates of the channel boundary surfaces.

According to the seventh assumption

$$\frac{\partial}{\partial z}\left(c\rho\omega_z T\right) >> \left[\frac{\partial}{\partial z}\left(\lambda\frac{\partial T}{\partial z}\right)\right] \quad \text{or} \quad \omega_z\frac{\partial T}{\partial z} >> a\frac{\partial^2 T}{\partial z^2}, \qquad /Ap.1.2/$$

where λ - is the liquid thermal conductivity. Thus it is possible to neglect the term $\frac{\partial^2 T}{\partial z^2}$ in the right side of /Ap.1.1/. The last assumption is realised with sufficient accuracy [24, 34] if $\frac{z}{2(r_2 - r_1)} >> \frac{1}{Pe}$, where $Pe = \bar{\omega}\cdot 2(r_2 - r_1)/a$ - is the Peclet number, $\bar{\omega}$ - is the mean velocity of the liquid. For example, for values of the Peclet number $Pe \geq 100$ the condition /Ap.1.2/ and consequently the assumption 7 are applicable beginning with $z=2(r_2-r_1)$ with an error of about 1%. Assumption 7 in practice is usually applicable for non-metallic liquids and gases (Prandtl number Pr=1...1000). In case of liquid metals (Pr=0.005...0.05) it can lose its applicability. In this case, for heat transfer solutions using equation /Ap.1.1/ it is necessary to include the term $a\frac{\partial^2 T}{\partial z^2}$.

i.Ap.1.1.1 Temperature Fields in Liquid Laminar Flows

Taking into account assumptions 1-7 the heat transfer problem in laminar flow in flat (r=0) or cylindrical (r=1) tubes with arbitrary initial conditions and boundary conditions of type k at $r=r_1$ and type m at $r=r_2$ (k, m=1, 2, 3) can be described as follows [24, 34, 55, 92-94]:

$$\omega_0\omega(r)\frac{\partial T(r,z)}{\partial z} - a\frac{1}{r^r}\frac{\partial}{\partial r}\left[r^r\frac{\partial T(r,z)}{\partial r}\right] = \frac{W(r,z)}{c\rho}, \; z>0,\; r_1<r<r_2, \qquad /Ap.1.3/$$

$$T(r,0) = T_b(r),\; l_k^1[T(r_1,z)] = f_k^1(z), \qquad l_m^2[T(r_2,z)] = f_m^2(z),$$

where T_b - is the temperature distribution at the tube inlet; ω_0 is the velocity value on the tube axis, $\omega(r)$ - is the velocity profile, f_k^1, f_m^2 - are functions, which are preset on the tube boundaries. The indices k, m determine the type of the boundary conditions at the points r_1 and r_2 and can accept independently the values 1, 2, 3.

The operators of the boundary conditions l_k^1 and l_m^2 have the form [24, 92]:

- when the boundary conditions are of the first type

$$l_1^s\left[T(r_s,z)\right] = T(r_s,z) = f_1^s(z), \; s = 1,2;$$

- when the boundary conditions are of the second type

$$l_2^s\left[T(r_s,z)\right]=(-1)^s\cdot\lambda\frac{\partial T(r_s,z)}{\partial r}=f_2^s(z),\ \ s=1,2;$$

- when the boundary conditions are of the third type

$$l_3^s\left[T(r_s,z)\right]=(-1)^s\cdot\frac{\lambda}{\alpha_s}\frac{\partial T(r_s,z)}{\partial r}+T(r_s,z)=f_3^s(z),\ \ s=1,2,$$

where α_1, α_2 - are the convective heat transfer coefficients on the surfaces $r=r_1$ and $r=r_2$.

The solution of /Ap.1.3/, which is obtained by the Green function method [37, 91-99] has the form [24, 52, 55, 92]:

$$T(r,z)=\int_{r_1}^{r_2}G(r,\xi,z,0)\xi^r\omega(\xi)T_b(\xi)d\xi+B_k^1(r,z)+B_m^2(r,z)+$$
$$+\int_0^z\int_{r_1}^{r_2}G(r,\xi,z,\eta)\xi^r\frac{W(\xi,\eta)}{\omega_0 c\rho}d\xi d\eta. \qquad /Ap.1.4/$$

The form of the functions $B_k^1(r,z)$ depend on the boundary conditions, which are preset at the point $r=r_1$ and the functions $B_m^2(r,z)$ depend on the type of boundary conditions at the point $r=r_2$:

$$B_1^s(r,z)=(-1)^{s+1}\frac{ar_s^r}{\omega_0}\int_0^z\frac{\partial G(r,r_s,z,\eta)}{\partial\xi}f_1^s(\eta)d\eta,\ \ s=1,2,$$

$$B_2^s(r,z)=(-1)^{s+1}\frac{ar_s^r}{\omega_0\lambda}\int_0^z G(r,r_s,z,\eta)f_2^s(\eta)d\eta,\ \ s=1,2,$$

$$B_3^s(r,z)=\frac{ar_s^r\alpha_s}{\omega_0\lambda}\int_0^z G(r,r_s,z,\eta)f_3^s(\eta)d\eta,\qquad s=1,2.$$

The Green function of the problem /Ap.1.3/

$$G(r,\xi,z,\eta)=\sum_{n=1}^{\infty}\frac{\psi_n\left(\frac{r}{r_2}\right)\psi_n\left(\frac{\xi}{r_2}\right)\exp\left[-\varepsilon_n^2\frac{a(z-\eta)}{\omega_0 r_2^2}\right]}{\int_{r_1}^{r_2}\psi_n^2\left(\frac{r}{r_2}\right)r^r\omega(r)dr}, \qquad /Ap.1.5/$$

where ε_n, $\psi_n(\bar{r})$ - are the eigenvalues and eigenfunctions of the Sturm-Liouville boundary problem [54, 193 - 196]:

$$\frac{d}{d\bar{r}}\left[\left(\bar{r}\right)^{\Gamma}\frac{d\psi\left(\bar{r}\right)}{d\bar{r}}\right]+\varepsilon^{2}\cdot\left(\bar{r}\right)^{\Gamma}\omega\left(\bar{r}\right)\psi\left(\bar{r}\right)=0,\quad \bar{r}=\frac{r}{r_2}, \qquad /Ap.1.6/$$

$$\overline{l_k^1}\left[\psi\left(\frac{r_1}{r_2}\right)\right]=0,\quad \overline{l_m^2}\left[\psi(1)\right]=0,\quad \frac{r_1}{r_2}<\bar{r}<1,$$

where $\overline{l_s^1}\left[\psi\left(\frac{r_s}{r_2}\right)\right]=\psi\left(\frac{r_1}{r_2}\right),\quad s=1,2,$

$$\overline{l_2^s}\left[\psi\left(\frac{r_s}{r_2}\right)\right]=(-1)^s\frac{d\psi\left(\frac{r_s}{r_2}\right)}{d\bar{r}},\quad s=1,2,$$

$$\overline{l_3^s}\left[\psi\left(\frac{r_s}{r_2}\right)\right]=(-1)^s\frac{\lambda}{\alpha_s r_s}\frac{d\psi\left(\frac{r_s}{r_2}\right)}{d\bar{r}}+\psi\left(\frac{r_s}{r_2}\right),\quad s=1,2.$$

For the flow of Newtonian and some non-Newtonian liquids in flat and round tubes the solution of /Ap.1.6/ is considered in [34-36, 54, 55]. In the general case, /Ap.1.6/ can be solved by numerical methods on a digital computer [56, 190]. The eigen values ε_n and eigen functions $\psi_n\left(\bar{r}\right)$ are given in [24, 34-36, 55].

i.Ap.1.1.2 Heat Transfer in Laminar Flow in a Flat Tube

Consider flow in a flat tube, i.e. between the two unbounded plates, which are at a distance h from each other. The origin of co-ordinates is placed on one of the tube surfaces. The axis z is directed along the flow and the axis r is directed perpendicularly to the wall.

Applying assumptions 1-7, r=0 and $\omega_0\omega(r)=6\bar{\omega}\left[\frac{r}{h}-\left(\frac{r}{h}\right)^2\right]$, where $\bar{\omega}$ - is the mean velocity, the problem is described as follows [24, 55]:

$$6\bar{\omega}\left[\frac{r}{h}-\left(\frac{r}{h}\right)^2\right]\frac{\partial T(r,z)}{\partial z}-a\frac{\partial^2 T(r,z)}{\partial r^2}=\frac{W(r,z)}{c\rho},\quad z>0,\quad 0<r<h,$$

$$T(r,0)=T_b(r),\ l_k^1\left[T(0,z)\right]=f_k^1(z),\ l_m^2\left[T(h,z)\right]=f_m^2(z). \qquad /Ap.1.7/$$

The indices k, m determine the type of the boundary conditions at the points 0 and h and can accept independently the values 1, 2, 3. The type of operators l_k^1, l_m^2 are given in i.Ap.1.1.1. The solution to /Ap.1.7/ has the form[24, 55]:

$$T(r,z)=\int_0^h G(r,\xi,z,0)\left[\frac{\xi}{h}-\left(\frac{\xi}{h}\right)^2\right]T_b(\xi)d\xi+$$
$$+\int_0^z\int_0^h G(r,\xi,z,\eta)\frac{W(\xi,\eta)}{6\overline{\omega}c\rho}d\xi d\eta+D_{km}(r,z), \qquad /Ap.1.8/$$

where $D_{km}(r,z)$, - is the function, which takes into account the influence of the boundary conditions at the points r=0 and r=h on the temperature field. The Green function is

$$G(r,\xi,z,\eta)=\chi_{km}+\sum_{n=1}^{\infty}\frac{\psi_n\left(\frac{r}{h}\right)\psi_n\left(\frac{\xi}{h}\right)\exp\left[-\varepsilon_n^2\frac{2a(z-\eta)}{3\omega_0 h^2}\right]}{\int_0^h\psi_n^2\left(\frac{\xi}{h}\right)\left[\frac{\xi}{h}-\left(\frac{\xi}{h}\right)^2\right]d\xi}, \qquad /Ap.1.9/$$

$$\chi_{km}=\begin{cases}6/h, k=m=2,\\ 0, \quad k\neq 2 \text{ or } m\neq 2.\end{cases}$$

The eigenvalues ε_n and the eigenfunctions ψ_n are determined by the Sturm-Liouville problem:

$$\frac{d^2\psi(\bar{r})}{d(\bar{r})^2}+\varepsilon^2\left[\bar{r}-(\bar{r})^2\right]\psi(\bar{r})=0, \quad \bar{r}=\frac{r}{h}, \quad 0<\bar{r}<1,$$

$$\bar{l}_k^1[\psi(0)]=0, \quad \bar{l}_m^2[\psi(1)]=0.$$

The type of operators $\bar{l}_k^1$ and $\bar{l}_m^2$ are given in i.Ap.1.1.1. The eigenfunctions $\psi_n(\bar{r})$ can be represented in the form of a power series [24, 55]:

$$\psi_n(\bar{r})=\sum_{i=0}^{\infty}b_i(\varepsilon_n)(\bar{r})^i,$$

$$b_i(\varepsilon_n)=\frac{4\varepsilon_n^2\left[b_{i-4}(\varepsilon_n)-b_{i-3}(\varepsilon_n)\right]}{i(i-1)}. \qquad /Ap.1.10/$$

The form of the function $D_{km}(r,z)$, the values of the first coefficients $b_i(\varepsilon_n)$, which are necessary for the calculation of the formula /Ap.1.10/ and also the type of the characteristic equations, the roots of which are the eigenvalues ε_n, depend on the boundary conditions and are given in Table Ap.1.1. The eigenvalues ε_n and the eigenfunctions $\psi_n(\bar{r})$ are reported in [24, 55].

Table Ap.1.1 Form of function $D_{km}(r,z)$

Boundary condition k	Boundary condition m	Form of function $D_{km}(r,z)$	Characteristic equations	$b_i(\varepsilon)$ $i=0,1,2,3,4$
1	2	3	4	5
1	1	$D_{11}=\frac{a}{6\varpi}\int_0^z\left\{\frac{\partial G(r,0,z-\eta)}{\partial\xi}f_1^1(\eta)-\frac{\partial G(r,h,z-\eta)}{\partial\xi}f_1^2(\eta)\right\}d\eta$	$\sum_{i=0}^{\infty}b_i(\varepsilon_n)=0$	$b_0(\varepsilon_n)=0$ $b_1(\varepsilon_n)=1$
1	2	$D_{12}=\frac{a}{6\varpi}\int_0^z\left\{\frac{\partial G(r,0,z-\eta)}{\partial\xi}f_1^1(\eta)+\frac{1}{\lambda}G(r,h,z-\eta)f_2^2(\eta)\right\}d\eta$	$\sum_{i=0}^{\infty}b_i(\varepsilon_n)\cdot i=0$	$b_2(\varepsilon_n)=0$ $b_3(\varepsilon_n)=0$
1	3	$D_{13}=\frac{a}{6\varpi}\int_0^z\left\{\frac{\partial G(r,0,z-\eta)}{\partial\xi}f_1^1(\eta)+\frac{\alpha_2}{\lambda}G(r,h,z-\eta)f_3^2(\eta)\right\}d\eta$	$\sum_{i=0}^{\infty}[b_i(\varepsilon_n)Bi_2+b_{i+1}(\varepsilon_n)(i+1)]=0$	$b_4(\varepsilon_n)=-\frac{\varepsilon_n^2}{3}$
2	1	$D_{21}=\frac{a}{6\varpi}\int_0^z\left\{\frac{1}{\lambda}G(r,0,z-\eta)f_2^1(\eta)-\frac{\partial G(r,h,z-\eta)}{\partial\xi}f_1^2(\eta)\right\}d\eta$	$\sum_{i=0}^{\infty}b_i(\varepsilon_n)=0$	$b_0(\varepsilon_n)=1$ $b_1(\varepsilon_n)=0$
2	2	$D_{22}=\frac{a}{6\varpi}\int_0^z\left\{\frac{1}{\lambda}G(r,0,z-\eta)f_2^1(\eta)+\frac{1}{\lambda}G(r,0,z-\eta)f_2^2(\eta)\right\}d\eta$	$\sum_{i=0}^{\infty}b_i(\varepsilon_n)\cdot i=0$	$b_2(\varepsilon_n)=0$
2	3	$D_{23}=\frac{a}{6\varpi}\int_0^z\left\{\frac{1}{\lambda}G(r,0,z-\eta)f_2^1(\eta)+\frac{\alpha_2}{\lambda}G(r,0,z-\eta)f_3^2(\eta)\right\}d\eta$	$\sum_{i=0}^{\infty}[b_i(\varepsilon_n)Bi_2+b_{i+1}(\varepsilon_n)(i+1)]=0$	$b_3(\varepsilon_n)=-\frac{2\varepsilon_n^2}{3}$

Table Ap. 1.1 (continued)

1	2	3	4	5
3	1	$D_{31}=\frac{a}{6\varpi}\int_0^z\left\{\frac{\alpha_1}{\lambda}G(r,0,z-\eta)f_3^1(\eta)-\frac{\partial G(r,h,z-\eta)}{\partial\xi}f_1^2(\eta)\right\}d\eta$	$\sum_{i=0}^{\infty}b_i(\varepsilon_n)=0$	$b_0(\varepsilon_n)=1$ $b_1(\varepsilon_n)=Bi_1$
3	2	$D_{32}=\frac{a}{6\varpi}\int_0^z\left\{\frac{\alpha_1}{\lambda}G(r,0,z-\eta)f_3^1(\eta)+\frac{1}{\lambda}G(r,h,z-\eta)f_2^2(\eta)\right\}d\eta$	$\sum_{i=0}^{\infty}b_i(\varepsilon_n)\cdot i=0$	$b_2(\varepsilon_n)=0$
3	3	$D_{33}=\frac{a}{6\varpi}\int_0^z\left\{\frac{\alpha_1}{\lambda}G(r,0,z-\eta)f_3^1(\eta)+\frac{\alpha_2}{\lambda}G(r,h,z-\eta)f_3^2(\eta)\right\}d\eta$	$\sum_{i=0}^{\infty}\left[b_i(\varepsilon_n)Bi_2+b_{i+1}(\varepsilon_n)(i+1)\right]=0$	$b_3(\varepsilon_n)=-\frac{2\varepsilon_n^2}{3}$

i.Ap.1.1.3 Heat Transfer in Laminar Flow in a Cylindrical Tube

Consider laminar flow in a round tube with the radius R. The origin of coordinates is on the tube axis. The axis z is directed along the flow and the axis r is directed along the tube radius. Taking into account assumptions 1-7, r=1 and $\omega_0\omega(r) = 2\bar{\omega}\left[1-\left(\frac{r}{R}\right)^2\right]$ the problem can be described as follows [24, 55]:

$$2\bar{\omega}\left[1-\left(\frac{r}{R}\right)^2\right]\frac{\partial T(r,z)}{\partial z} - a\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial T(r,z)}{\partial r}\right] = \frac{W(r,z)}{c\rho}, \qquad /Ap.1.11/$$

$$z>0,\ 0<r<R,$$

$$T(r,0)=T_b(r), \quad \frac{\partial T(0,z)}{\partial r}=0, \quad l_j^2[T(R,z)] = f_j^2(z).$$

The index j determines the type of the boundary condition at the point r=R and can accept the values 1, 2, 3. The type of the operator l_j^2 is considered in i.Ap.1.1.1.

The solution T(r,z) of /Ap.1.11/ has the form [24, 55]:

$$T(r,z) = \int_0^R G(r,\xi,z,0)\xi\left[1-\left(\frac{\xi}{R}\right)^2\right]T_b(\xi)d\xi + \qquad /Ap.1.12/$$
$$+\int_0^z\int_0^R G(r,\xi,z,\eta)\xi\frac{W(\xi,\eta)}{2\bar{\omega}\cdot c\rho}d\xi d\eta + F_j(r,z),$$

where $F_j(r,z)$ - is the function, which takes into account the influence of the boundary conditions at the point r=R on the temperature field.

The Green function of the problem /Ap.1.11/ is

$$G(r,\xi,z,\eta) = \chi_j + \sum_{n=1}^{\infty}\frac{\psi_n\left(\frac{r}{R}\right)\psi_n\left(\frac{\xi}{R}\right)\exp\left[-\varepsilon_n^2\frac{a(z-\eta)}{2\bar{\omega}R^2}\right]}{\int_0^R\psi_n^2\left(\frac{r}{R}\right)r\left[1-\left(\frac{r}{R}\right)^2\right]dr}, \qquad /Ap.1.13/$$

$$\chi_j = \begin{cases}4/R^2, & j=2,\\ 0, & j\neq 2.\end{cases}$$

The eigen values ε_n and the eigen functions ψ_n are determined by the Sturm-Liouville problem:

$$\frac{d}{d\bar{r}}\left[\bar{r}\frac{d\psi(\bar{r})}{d\bar{r}}\right]+\varepsilon^2\bar{r}\left[1-(\bar{r})^2\right]\psi(\bar{r})=0, \quad \bar{r}=\frac{r}{R}, \quad 0<\bar{r}<1,$$

$$\frac{d\psi(0)}{d\bar{r}}=0, \quad \bar{l}_j^{-2}\left[\psi(1)\right]=0,$$

the solution of which can be recorded in the form of a power series:

$$\psi(\bar{r})=\sum_{i=0}^{\infty} b_{2i}(\varepsilon_n)(\varepsilon_n\bar{r})^{2i},$$

$$b_{2i}(\varepsilon_n)=\frac{1}{(2i)^2}\left[\frac{b_{2i-4}(\varepsilon_n)}{\varepsilon_n^2}-b_{2i-2}(\varepsilon_n)\right]. \quad /Ap.1.14/$$

The form of the function $F_j(r,z)$, the values of the first two coefficients $b_0(\varepsilon_n)$ and $b_2(\varepsilon_n)$, necessary for calculations from /Ap.1.14/ and also the characteristic equations for the calculation of the eigenvalues ε_n, depend on the series of the boundary conditions on the tube surface at r=R and are given in Table Ap.1.2.

Table Ap.1.2 Form of the Function $F_j(r,z)$, which Determines the Dependence of the Solution of /Ap.1.12/ on Boundary Conditions

Boundary condition	Form of function $F_j(r,z)$	Characteristic equations	$b_i(\varepsilon_n)$
1	$F_1(r,z)=$ $=\frac{a}{2\varpi}\int_0^z\frac{\partial G(r,R,z,-\eta)}{\partial\xi}Rf_1^2(\eta)d\eta$	$\sum_{i=0}^{\infty} b_{2i}(\varepsilon_n)\cdot(\varepsilon_n)^{2i}=0$	
2	$F_2(r,z)=$ $=\frac{a}{2\varpi\lambda}\int_0^z G(r,R,z-n)Rf_2^2(\eta)d\eta$	$\sum_{i=0}^{\infty} b_{2i}(\varepsilon_n)\cdot 2i(\varepsilon_n)^{2i}=0$	$b_0(\varepsilon_n)=$ $=1$
3	$F_3(r,z)=$ $=\frac{a\alpha}{2\varpi\lambda}\int_0^z G(r,R,z-n)Rf_3^2(\eta)d\eta$	$\sum_{i=0}^{\infty} b_{2i}(\varepsilon_n)\cdot(2i+$ $+Bi_2)(\varepsilon_n)^{2i}=0$	$b_2(\varepsilon_n)=$ $=-\frac{1}{4}$

The eigen numbers ε_n and eigen functions $\psi_n(\bar{r})$, to calculate the heat transfer in a round cylindrical tube, are given in [24, 34, 55].

It is possible to approximate the Green functions by a finite series /Ap.1.5/, /Ap.1.9/, /Ap.1.13/ with the error set in advance. The equations /Ap.1.4/, /Ap.1.8/, /Ap.1.12/ can easily be programmed on a digital computer and used as the algorithms of the different problems solutions of liquid heating [24, 55] with changes of temperatures, heat fluxes, thermophysical properties, flow velocities, tubes dimensions and other parameters.

It is of interest to note that the solutions can also be used for the calculations of mass transfer in laminar flows of a two-component liquid in tube flow, because in many cases the corresponding problem can be described in the form of /Ap.1.3/, /Ap.1.7/, /Ap.1.11/. The more complicated cases of the calculation of heat- and (or) mass transfer in liquid laminar flows, based on the use of Greens functions, are considered in [94, 96, 97, 163, 164, 172, 173, 179, 197]. Calculations of temperature fields in liquid turbulent flows [184 - 189], based on the use of Greens functions, are published in [24, 52, 78]. Algorithm of apparent turbular Prandtl number calculation is considered in [223].

§Ap.1.2 Calculation of Temperature Pattern in Multilayer System in Form of Two Coaxially Mounted Cylinders with Liquid in Clearance between them

Ap.1.2.1 Use of Green Function Method for Multilayer System Temperature Pattern Calculation

The physical model of the considered system consists of two coaxially mounted cylinders with a liquid placed in the clearance between them. The circuit of this system is analogous to that one given in Fig.4.1. This system consists of n layers. The inner cylinder B is in the form of (n-2) layers. The outer cylinder H makes layer number n and is capable of rotating with angular frequency ω. The temperature of its outer surface can vary in time with the arbitrary law $T(R_n, \tau) = T_r(\tau)$. The clearance between the cylinders B and H makes the (n-1) layer which can be filled with the liquid being investigated. On the boundaries between the layers in the coaxial cylindrical sections at $r = R_j$, $j = 1, 2, ..., n-1$ surface heat sources can act with the specific power P_j, $j = 1, 2, ..., n-1$. In the multilayer system being considered internal heat sources having the volumetric density $W(r, \tau)$ can act. Let us also note that in the layer (n-1) of the liquid being investigated the internal heat sources can act

$$W(r) = \frac{\mu 4\omega^2 R_{n-1}^4 R_{n-2}^4}{c_{n-1}\rho_{n-1}\left(R_{n-1}^2 - R_{n-2}^2\right)^2 r^4}, \quad R_{n-2} < r < R_{n-1}$$

where μ - is the dynamic viscosity of the liquid in the clearance between the stationary inner cylinder B and the outer cylinder H. This outer cylinder rotates with the angular frequency ω. Let us denote the thermophysical properties within each layer as c_j, ρ_j, λ_j, $a_j = \frac{\lambda_j}{c_j \rho_j}$ which are correspondingly the specific heat, density, thermal conductivity and thermal diffusivity of the material of the j layer.

Taking into account these designations the temperature pattern $T(r, \tau)$ of the considered n layer system at the point with the radial coordinate r in the moment of time τ is modeled by the differential equation of the thermal conduction [176]:

$$c(r)\rho(r)\frac{\partial T(r,\tau)}{\partial \tau} = \frac{1}{r^{r}}\frac{\partial}{\partial r}\left[r^{r}\lambda(r)\frac{\partial T(r,\tau)}{\partial r}\right] + W(r,\tau), \qquad /Ap.1.15/$$

$$\tau > 0,\ 0 < r < R_n,\ 0 < R_1 < R_2 < \ldots < \ldots < R_n,$$

$$c(r) = c_j \text{ at } R_{j-1} < r < R_j,\ \rho(r) = \rho_j \text{ at } R_{j-1} < r < R_j,$$

$$\lambda(r) = \lambda_j \text{ at } R_{j-1} < r < R_j,\ W(r, \tau) = W_j(r, \tau) \text{ at}$$

$$R_{j-1} < r < R_j,\ j = 1, 2, \ldots, n,\ R_o = 0,$$

with the boundary conditions:

$$\frac{\partial T(0,\tau)}{\partial r} = 0,$$

$$T(R_j - 0, \tau) = T(R_j + 0, \tau),\ j = 1, 2, \ldots, n\text{-}1, \qquad /Ap.1.16/$$

$$\lambda_j \frac{\partial T(R_j - 0,\tau)}{\partial r} - \lambda_{j+1}\frac{\partial T(R_j + 0,\tau)}{\partial r} = P_j(\tau),\ \ j = 1, 2, \ldots, n\text{-}1,$$

$$T(R_n, \tau) = T_r(\tau),$$

and with the initial condition $T(r, 0) = T_b(r)$.

In the problems for the flat, cylindrical and spherical coordinates systems in the thermal conduction equation the coefficient of the form r is used (r = 0, 1, 2, correspondingly for the flat, cylindrical and spherical coordinate system).

The analytical solution of this problem which was obtained with the use of the Green function method has the form:

$$\begin{aligned} T(r,\tau) = & \int_0^{R_n} G(r,\xi,\tau)\xi^r c(\xi)\rho(\xi)T_b(\xi)d\xi + \\ & + \int_0^{\tau}\int_0^{R_n} G(r,\xi,\tau-\eta)\xi^r W(\xi,\eta)d\xi d\eta - \\ & - \int_0^{\tau} \frac{\partial G(r,R_n,\tau-\eta)}{\partial\xi}\lambda_n R_n^r T_r(\eta)d\eta + \\ & + \sum_{j=1}^{n-1}\int_0^{\tau} G(r,R_j,\tau-\eta)R_j^r P_j(\eta)d\eta, \end{aligned} \qquad /Ap.1.17/$$

where

$$G(r,\xi,\tau-\eta) = \sum_{i=1}^{\infty} \frac{\psi_i(r)\psi_i(\xi)\exp\left[-\varepsilon_i^2 \frac{a(\tau-\eta)}{R_n^2}\right]}{\int_0^{R_n}\psi_i^2(x)x^r c(x)\rho(x)dx} \qquad /Ap.1.18/$$

is the Green function of the considered boundary problem, ε_i, $\psi_i(r)$ are the eigen values and eigen functions of the Sturm-Liouville boundary problem

$$\frac{d}{dr}\left[r^r\bar{\lambda}(r)\frac{d\psi(r)}{dr}\right] + \varepsilon^2 r^r \bar{c}(r)\bar{\rho}(r)\psi(r) = 0\,, \quad 0 < r < R_n,$$

$$\bar{\lambda}(r) = \frac{\lambda(r)}{\lambda}, \qquad \bar{c}(r) = \frac{c(r)}{c}, \quad \bar{\rho}(r) = \frac{\rho(r)}{\rho}, \quad a = \frac{\lambda}{c\rho}, \qquad /Ap.1.19/$$

$$\frac{d\psi(0)}{dr} = 0, \quad \psi(R_j - 0) = \psi(R_j + 0), \qquad j = 1,2,\ldots,n-1\,,$$

$$\lambda_j \frac{d\psi(R_j - 0)}{dr} = \lambda_{j+1}\frac{d\psi(R_j + 0)}{dr}, \quad j = 1,2,\ldots,n-1; \quad \psi(R_n) = 0\,,$$

which arises at the solution of the initial problem /Ap.1.15/, /Ap.1.16/.

For the solution using for the mathematical modeling of the thermal processes in the measuring device being considered in chapter 4, it is convenient to use the thermophysical properties of the (n-2) or (n-1) layers as the values c, ρ, λ, $a = \frac{\lambda}{c\rho}$, in the Green function /Ap.1.18/ and of the Sturm-Liouville boundary problem /Ap.1.19/. For example:

$$c = c_{n-2},\quad \rho = \rho_{n-2},\quad \lambda = \lambda_{n-2},\quad a = a_{n-2} = \frac{\lambda_{n-2}}{c_{n-2}\rho_{n-2}}$$

or

$$c = c_{n-1},\quad \rho = \rho_{n-1},\quad \lambda = \lambda_{n-1},\quad a = a_{n-1} = \frac{\lambda_{n-1}}{c_{n-1}\rho_{n-1}}.$$

The solution /Ap.1.17/ was used in the initial stage for the calculation of the temperature patterns of the device for the investigation of the dependence of the thermal conductivity and thermal diffusivity tensors second diagonal components on the shear rate which is considered in chapter 4. Examples of other cases of heat or mass transfer modeling in solids are considered in [24, 52, 166, 176, 212 - 217].

§Ap. 1.2.2 Algorithm of Numerical Problem Solution of Calculation of Temperature Pattern in Multilayer System in Form of Two Coaxially Mounted Cylinders with Liquid in Clearance between them*

Let us consider an N -layer cylinder of infinite length with the inner radius $R_o > 0$ and outer radius R_N. We shall assume that the temperature of T_j layer depends only on time and radius, i.e. T = T(r, τ), τ is the time, r - is the radius. In this case the temperature pattern in the j layer satisfies the equation:

$$c_j\rho_j\frac{\partial T}{\partial \tau} = \frac{1}{r}\frac{\partial}{\partial r}\left(\lambda_j r\frac{\partial T}{\partial r}\right) + W_j(r,\tau),$$

$$\tau > 0,\quad R_{j-1} < r < R_j,\quad 1 \le j \le N, \qquad \text{/Ap.1.20/}$$

where c_j - is the specific heat; ρ_j - is the density, λ_j - is the thermal conductivity of the j layer; W_j - is the amount of heat released per unit volume and time.

The equations /Ap.1.20/ are completed by the boundary conditions on the inner

$$\alpha_L\frac{\partial T}{\partial r} + \beta_L T\Big|_{r=R_O} = \varphi_L(\tau) \qquad \text{/Ap.1.21/}$$

* - numerical scheme and program were worked out together with Dr. A.I. Urusov

and outer

$$\alpha_R \frac{\partial T}{\partial r} + \beta_R T\Big|_{r=R_N} = \varphi_R(\tau) \qquad /Ap.1.22/$$

surfaces of the cylinder with the initial condition

$$T\Big|_{\tau=0} = T_b(r), \qquad R_o \le r \le R_N, \qquad /Ap.1.23/$$

and also by the inner boundary conditions which arise from the requirement of the continuity of the temperature and heat fluxes. In general, to assume the possibility of the existence of sources (or sinks) on the borderlines between the layers then it is possible to record the inner boundary conditions in the form

$$T\Big|_{r=R_j-0} = T\Big|_{r=R_j+0}, \qquad 1 \le j \le N-1, \qquad /Ap.1.24/$$

$$\lambda_j \frac{\partial T}{\partial r}\Big|_{r=R_j-0} - \lambda_{j+1} \frac{\partial T}{\partial r}\Big|_{r=R_j+0} = P_j(\tau), \qquad 1 \le j \le N-1, \qquad /Ap.1.25/$$

The equations system /Ap.1.20/ - /Ap.1.25/ is a closed system of equations for the temperature patterns determination at any time. For the fully developed temperature field determination, it is possible to use the difference diagram, described below, for the solution of this problem. This is stipulated by the fact that equation /Ap.1.20/ is a parabolic type and at $\tau = \infty$ the temperature distribution which is received from problem /Ap.1.20/ - /Ap.1.25/ solution leads to the solution of the corresponding steady state problem, i.e. as a result we receive the steady state temperature distribution in the multilayer cylinder.

To apply the finite difference method to the problem /Ap.1.20/ - /Ap.1.25/, the following was taken into consideration: the use of an explicit formulation is not expedient here because of the stability constraint conditions which don't allow for an effective solution. In order to have the opportunity to solve the steady state problem the method must have absolute stability and consequently it is necessary to develop an implicit formulation and this leads to the necessity to solve a system of algebraic equations which can be solved by a simple and rapid (in sense of machine time) algorithm. Taking into account the above we come to the conclusion that for the problem /Ap.1.20/ - /Ap.1.25/ it is necessary to utilize a finite difference formulation which is possible to realize by a three-point formulation.

Let us assume that n_j - is a number of intervals on the length $[R_{j-1}, R_j]$, $1 \le j \le N$, $h_j = \dfrac{R_j - R_{j-1}}{n_j}$ - is the distance between the nodes of the net in the j layer, t - is the spacing, $r_k^j = R_{j-1} + kh_j$, $0 \le k \le n_j$, $T_{jk}^m = T(mt, r_k^j)$.

With these designations, we specify the finite difference scheme for the equation /Ap.1.20/ in the form (we assume that λ_j = const for each layer):

$$c_j\rho_j \frac{T_{jk}^{m+1} - T_{jk}^{m}}{t} = \frac{\lambda_j}{h_j^2}\left\{ \frac{r_{k+1/2}^{j}}{r_k^j} T_{j,k+1}^{m+1} - 2T_{j,k}^{m+1} + \right.$$

$$\left. + \frac{\lambda_{k-1/2}^{j}}{r_k^j} T_{j,k-1}^{m+1} \right\} + W_{j,k}^{m+1}, \quad 1 \le j \le N, \quad 1 \le k \le n_j - 1 \qquad \text{/Ap.1.26/}$$

The approximation of the boundary conditions on the inner and outer cylinder surfaces depends on the coefficients α_L, β_L, α_R, β_R:

if $\left|\beta_L - \frac{\alpha_L}{h_1}\right| < \left|\frac{\alpha_L}{h_1}\right|$

$$\alpha_L \frac{T_{1,1}^{m} - T_{1,0}^{m}}{h_1} + \beta_L T_{1,0}^{m+1} = \varphi_L^{m+1}, \qquad \text{/Ap.1.27/}$$

if $\left|\beta_L - \frac{\alpha_L}{h_1}\right| \ge \left|\frac{\alpha_L}{h_1}\right|$

$$\alpha_L \frac{T_{1,1}^{m+1} - T_{1,0}^{m+1}}{h_1} + \beta_L T_{1,0}^{m+1} = \varphi_L^{m+1}, \qquad \text{/Ap.1.27a/}$$

if $\left|\beta_R + \frac{\alpha_R}{h_N}\right| < \left|\frac{\alpha_R}{h_N}\right|$

$$\alpha_R \frac{T_{N,n_{N-1}}^{m} - T_{N,n_{N-1}}^{m}}{h_N} + \beta_R T_{N,n_N}^{m+1} = \varphi_R^{m+1}, \qquad \text{/Ap.1.28/}$$

if $\left|\beta_R + \frac{\alpha_R}{h_N}\right| \ge \left|\frac{\alpha_R}{h_N}\right|$

$$\alpha_R \frac{T_{N,n_N}^{m+1} - T_{N,n_{N-1}}^{m+1}}{h_N} + \beta_R T_{N,n_N}^{m+1} = \varphi_R^{m+1}. \qquad \text{/Ap.1.28a/}$$

The approximation of the internal boundary conditions has the form:

$$T_{j,n_j}^{m+1} = T_{j+1,0}^{m+1}, \quad 1 \le j \le N-1, \qquad \text{/Ap.1.29/}$$

$$\lambda_j \frac{T_{j,n_j}^{m+1} - T_{j,n_{j-1}}^{m+1}}{h_j} - \lambda_{j+1} \frac{T_{j+1,1}^{m+1} - T_{j+1,0}^{m+1}}{h_{j+1}} = P_j^{m+1}, \qquad /Ap.1.30/$$

$$1 \le j \le N-1 .$$

When making the computing program on the basis of the above finite difference scheme, the method of a three- point formulation [123 - 128] was used. Other algorithms of temperature patten numerical calculation in laminar flows and in solids were considered in [205, 206, 212 - 217].